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AN

ELEMENTARY TREATISE

ON

PLANE AND SOLID GEOMETRY

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PREFACE.

The use of *infinitely small quantities*, which was first introduced into the higher departments of Mathematics, has been gradually creeping downwards, and elementary writers are rapidly becoming reconciled to it. But at the same time, the uncompromising advocates of the ancient rigor of demonstration have, by their attacks, induced some mathematicians to waste much time in disguising the principles of the Differential Calculus under a form of words, in which the term "*infinitely small*" does not occur. The value of this labor may be duly estimated from the inconsistency of one, who has ostensibly discarded the infinitesimal doctrine from his theory of the Calculus, and introduced it into his treatise of Geometry. With all its boasted rigor, the ancient Geometry can indeed lead to no result more accurate, none more to be depended upon, than those of the infinitesimal theory; and I doubt if any well constituted mind, well constituted at least for mathematical investigations, ever reposes with any more confidence upon the one than upon the other. If there were

any error involved in the latter theory, it must not only be infinitely small, but must remain infinitely small after all the magnifying processes to which it could possibly be subjected. But there is no error; for, if we suppose that there be an error which we may represent by $\mathcal{A}$, since the aggregate of all the quantities neglected in arriving at the result is infinitely small, that is, as small as we choose, we may choose it to be smaller than $\mathcal{A}$; and, therefore, the error $\mathcal{A}$ is greater than the greatest possible error which could be obtained, a manifest absurdity, but one which cannot be avoided as long as $\mathcal{A}$ is any thing.

The term *direction* is introduced into this treatise without being defined; but it is regarded as a simple idea, and to be as incapable of definition as *length*, *breadth*, and *thickness*; and this innovation will probably be pardoned, when it is seen how much it contributes to the brevity and simplicity of demonstration, which I have everywhere studied.

BENJAMIN PEIRCE

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EXPLANATION OF SIGNS,

AND

OF SOME USEFUL PROPOSITIONS IN THE DOCTRINE OF PROPORTIONS.

The sign $+$ is *plus*, or *added to*. Thus $A + B$ is A added to B.

The sign $-$ is *minus*, or *less*. Thus, $A - B$ is A less B.

The sign $\times$ is *multiplied by*. Thus, $A \times B$ is A multiplied by B; and the period (.) is also the sign of multiplication.

The sign $\div$ or : is *divided by*. Thus, $A \div B$ or $A : B$ is A divided by B. The quotient of A divided by B may also be written $\frac{A}{B}$.

The sign $=$ is *equal to*. Thus, $A = B$ is A equal to B; and the expression in which this sign occurs is called an *equation*.

The sign $>$ is *greater than*. Thus, $A > B$ is A greater than B.

The sign $<$ is *less than*. Thus, $A < B$ is A less than B.

A^2 indicates the *second power* of A, A^3 the *third power*, &c.

A *ratio* or *fraction* is the quotient of one quantity divided by another, and is usually written with the sign (:). Thus the ratio of A to B is $A : B$, or it may just as well be written in the form of a fraction, as $\frac{A}{B}$.

The first term of a ratio is called the *antecedent*, and

the second the *consequent*. Thus, A is the antecedent of the preceding ratio, and B its consequent.

The value of a ratio is not altered by multiplying or dividing both its terms by the same number. Thus, $A : B$ is equal to $m \times A : m \times B$.

A *proportion* is the equation formed by two equal ratios Thus, if the two ratios $A : B$ and $C : D$ are equal, the equation $$A : B = C : D$$
is a proportion, and it may also be written
$$\frac{A}{B} = \frac{C}{D}.$$

The first and last terms of a proportion are called its *extremes*; and the second and third its *means*. Thus, A and D are the extremes of this proportion, and B and C its means.

Theorem I. The product of the means of a proportion is equal to the product of its extremes.

Proof. If the fractions of a proportion
$$A : B = C : D$$
are reduced to a common denominator, they give
$$\frac{A \times D}{B \times D} = \frac{B \times C}{B \times D}$$
or, omitting the common denominator,
$$A \times D = B \times C.$$
This proposition is called the *test* of proportions.

Theorem II. If four quantities are such that the product of the first and last of them is equal to the product of the second and third, these four quantities form a proportion

Proof. Let A, B, C, D, be such that
$$A \times D = B \times C.$$
Dividing by $B \times D$ we have
$$\frac{A \times D}{B \times D} = \frac{B \times C}{B \times D};$$

which, reduced to lower terms, and written in the form of ratios, is

$$A : B = C : D.$$

Corollary. The terms of a proportion may be transposed in any way, provided the product of the means is retained equal to that of the extremes, and the proportion will not be destroyed.

Thus, the preceding proportion gives, by transposition,

$$A : C = B : D,$$
$$B : A = D : C,$$
$$B : D = A : C, \text{ \&c.}$$

If both the means of the proportion are of the same magnitude, this mean is called the *mean proportional* between the extremes. Thus, if

$$A : B = B : D,$$

B is a mean proportional between A and D.

Theorem III. The mean proportional between two quantities is the square root of their product.

Proof. The application of the test to the preceding proportion gives

$$B^2 = A \times D,$$

the square root of which is

$$B = \angle (A \times D).$$

A succession of several equal ratios is called a *continued proportion.* Thus,

$$A . B = C : D = E : F = \text{\&c.}$$

is a continued proportion.

Theorem IV. The sum of any number of antecedents of a continued proportion is to the sum of the corresponding consequents as one antecedent is to its consequent.

Proof. Denote the common value of the ratios in the above continued proportion by M, we have

$$M = A : B = C : D = \&c \;;$$

whence

$$A = B \times M,$$
$$C = D \times M,$$
$$E = F \times M, \&c.$$

and the sum of these equations is

$$A + C + E + \&c. = (B + D + F + \&c.) \times M\,;$$

whence

$$\frac{A + C + E + \&c.}{B + D + F + \&c.} = M = \frac{A}{B} = \frac{C}{D} = \&c.$$

Corollary. The sum of the antecedents of a proportion is to the sum of its consequents as either antecedent is to its consequent; and the difference of the antecedents is to the difference of the consequents in the same ratio.

Theorem V. The sum of the antecedents of a proportion is to their difference, as the sum of the consequents is to their difference.

Proof. The proportion

$$A : B = C : D$$

gives, by the preceding proposition,

$$A + C : B + D = A - C : B - D$$

whence, by transposing the means,

$$A + C : A - C = B + D : B - D.$$

Theorem VI. The sum of the first two terms of a proportion is to the sum of the last two as the first term is to the third, or as the second is to the fourth; and the difference of the first two terms is to the difference of the last two in the same ratio; also the sum of the first two terms is to their difference as the sum of the last two is to their difference

Proof. The proportion

$$A : B = C : D$$

gives, by transposing the means,

$$A : C = B : D;$$

from which we obtain, by the preceding propositions,

$$A + B : C + D = A - B : C - D = A : C = B : D$$
$$A + B : A - B = C + D : C - D.$$

Two proportions, as

$$A : B = C : D$$

and

$$E : F = G : H,$$

may evidently *be multiplied together term by term, and the result*

$$A \times E : B \times F = C \times G = D \times H$$

is a new proportion.

Likewise, a proportion may be multiplied by itself any number of times in succession, *and the squares, cubes, fourth powers, &c. of the terms form a new proportion.* Thus, the proportion

$$A : B = C : D$$

gives

$$A^2 : B^2 = C^2 : D^2$$
$$A^3 : B^3 = C^3 : D^3$$
$$A^4 : B^4 = C^4 : D^4, \text{ \&c. \&c.}$$

GEOMETRY.

CHAPTER I.

GENERAL REMARKS AND DEFINITIONS.

1. *Definition.* *Geometry* is the Science of *Position* and *Extension.*

2. *Definition.* A *Point* has merely position, without any extension.

3. *Definition.* *Extension* has three dimensions: *Length*, *Breadth*, and *Thickness.*

4. *Definition.* A *Line* has only one dimension, namely, length.

5. *Definition.* A *Surface* has two dimensions; length and breadth.

6. *Definition.* A *Solid* has the three dimensions of extension; length, breadth, and thickness.

7. *Scholium.* The boundaries of solids are surfaces, the limits of surfaces are lines, and the extremities of lines are points.

The Point, then, on account of its simplicity, deserves our first consideration.

CHAPTER II.

THE POINT.

8. The *Position* of a Point is determined by its *Direction* and *Distance* from any known point; in other words, the *Elements* of its Position are Direction and Distance.

Remarks. The Direction of a Point is readily ascertained without any change in the position of the observer, whereas the determination of its distance is often more difficult, as it requires some change of place proportionate to the distance to be measured; thus, the direction of a star is seen at a glance, while the most profound science and the most accurate observations have not enabled the astonomer to ascertain its distance.

9. The *Direction* of a Point from the observer may be determined by a reference to some known direction, such as that of the zenith, the pole-star, &c.

The method by which one direction may thus be referred to another will be more definitely treated of in a succeeding article.

10. The *Distance* of a Point from the observer is the *length* of the shortest line drawn to the point; and it may be determined by a reference to some known length, such as an inch, a yard, a metre, a mile, &c.

CHAPTER III.

THE STRAIGHT LINE.

11. *Definition.* The *Direction of a Line* in any part is the direction of a point at that part from the next preceding point of the line.

a. Thus the direction of the line *AB* (fig 1) at *P* is the same as the direction of *P* from *O*.

b. In the same way, the direction of the line at *P* is the same as that of *O* from *P*, or the *opposite* direction to the preceding; and, consequently, a line has two different directions exactly opposed to each other, either of which may be *assumed* as the direction of the line.

12. *Definition.* A *Straight line* is one, the direction of which is the same throughout, as *AB* (fig. 2).

13. *Definitions.* A *Broken or Polygonal Line* is one, which is composed of straight lines, as *ABCD* (fig. 3).

A *Curved Line* is one, the direction of which is constantly changing, as AB (fig. 1).

14. *Definition.* A *Plane* is a surface in which any two points being taken, the straight line joining those points lies wholly in that plane.

15. *Axiom.* The direction of any point of a straight line from any preceding point, is the same as the direction of the line itself.

Thus the direction of *P* or *B* (fig. 2) from *M* or *A* is the same as that of the line *AB*

16. *Theorem.* The position of a straight line is determined by means of two points.

For, by the preceding axiom, these two points determine its direction.

17. *Theorem.* All the points which lie in the same direction from a given point are in the same straight line.

Proof. Thus, if P and M (fig. 2) are in the same direction from A, the two straight lines AP and AM must likewise, by § 15, have the same direction, and must consequently coincide in the same straight line.

18. *Axiom.* A straight line is the shortest way from one point to another.

CHAPTER IV.

THE ANGLE.

19. *Definitions.* An *Angle* is formed by two lines meeting or crossing each other.

The *Vertex* of the angle is the point where its *sides* meet.

The *magnitude* of the angle depends solely upon the *difference of direction* of its sides at the vertex.

a. The magnitude of the angle does not depend upon the length of its sides. Thus the angle formed by the two lines AB and AC (fig. 4) is not changed by shortening or lengthening either or both of these lines.

b. The method of denoting the angle is by the three letters BAC, the letter A which is at the vertex being placed in the *middle*; or the letter A may be used by itself, when this can be done without confusion.

20. *Definition*. When one straight line meets or crosses another, so as to make the two adjacent angles equal, each of these angles is called a ***Right angle***, and the lines are said to be ***perpendicular*** to each other.

Thus the angles ABC and ABD (fig. 5), being equal, are right angles.

21. *Definitions*. An *Acute* angle is one less than a right angle, as A (fig. 4).

An *Obtuse* angle is one greater than a right angle, as A (fig. 6).

22. *Definitions*. The *Complement* of an angle is the remainder, after subtracting it from a right angle.

The *Supplement* of an angle is the remainder, after subtracting it from two right angles.

23. *Theorem*. When one straight line meets or crosses another, the two adjacent angles are supplements of each other, and the *vertical* angles are equal to each other.

Proof. Let AB and CD (fig. 7) be the two lines. The adjacent angles APC and APD are supplements, for, if the perpendicular PM be erected, we have, by inspection,

$$\begin{aligned} APC + APD &= MPC + MPD \\ &= \text{two right angles.} \end{aligned}$$

b. In the same way, APC and BPC may be proved to be supplements of each other; and therefore the vertical

angles APD and BPC must be equal, since they have the same supplement APC.

In the same way, it may be shown that the vertical angles APC and BPD are equal.

c. Corollary. If either of the angles APC, APD, BPC, or BPD is a right angle, the other three must also be right angles.

d. Scholium. As a straight line has two different directions exactly opposed to each other, it is not unfrequently considered as making an angle with itself equal to two right angles.

24. *Corollary.* If the two adjacent angles APC and APD (fig. 8) are supplements of each other, their exterior sides PC and PD must be in the same straight line.

25. *Theorem.* The sum of all the successive angles APB, BPC, CPD, DPE (fig. 9), formed in a plane on the same side of a straight line AE, is equal to two right angles.

Proof. For it is equal to the sum of the two right angles APM, MPE, formed by the perpendicular PM.

26. *Theorem.* The sum of all the successive angles APB, BPC, CPD, DPE, and EPA (fig. 10), formed in a plane about a point, is equal to four right angles.

Proof. For it is equal to the sum of the four right angles MPN, NPM', $M'PN'$, $N'PM$, formed by the two perpendiculars MM' and NN'.

CHAPTER V.

PARALLEL LINES.

27. *Definition.* *Parallel* Lines are straight lines which have the same Direction, as AB, CD (fig. 11).

28. *Theorem.* Parallel lines cannot meet, however far they are produced.

Proof Thus the two lines AB and CD (fig. 11) cannot meet at P; for, if two straight lines are drawn through P, in the same direction, they must coincide and form one and the same straight line.

29. *Theorem.* Two angles, as A and B (fig. 12), are equal, when they have their sides parallel and directed the same way from the vertex.

Proof. For, as the directions of BD and BF are respectively the same as those of AC and AE, the difference of direction of BD and BF must be the same as that of AE and AC; that is, by § 19 the angle A is equal to the angle B.

30. *Theorem.* If two parallel lines AB, CD (fig. 13) are cut by a third straight line EF, the *external-internal* angles, as EMB and END, or BMF and DNF, are equal, and the *alternate-internal* angles, as AMN and MND, or BMN and MNC, are also equal.

Proof. *a.* The external-internal angles are equal, because their sides have the same direction

Angles made by a Line cutting Parallel Lines.

b. The alternate-internal angles are equal, as *AMN* and *MND* because *AMN* is, by § 23, equal to its vertical angle *EMB*, which has just been proved equal to *MND*.

31. *Theorem.* If two straight lines, lying in the same plane, as *AB*, *CD* (fig. 13), are cut by a third, *EF*, so that the angles *EMB* and *END* are equal, or *AMN* and *MND* are equal, &c.; the lines *AB*, *CD* must be parallel.

Proof. For the line, drawn through the point *M* parallel tc *CD*, must make these angles equal, and must therefore coincide with *AB*.

32. *Theorem.* If two parallel lines *AB*, *CD* (fig. 13) are cut by a third straight line *EF*, the two interior angles on the same side, as *BMN* and *MND*, are supplements of each other.

Proof. For *BMN* is, by § 23, the supplement of its adjacent angle *EMB*, which is equal to *MND*.

33. *Theorem.* If two straight lines, lying in the same plane, as *AB* and *CD* (fig. 13), are cut by a third, *EF*, so that the angles *BMN* and *MND* are supplements of each other, the lines *AB*, *CD* must be parallel.

Proof. For the line, drawn through the point *M* parallel tc *CD*, must make these angles supplements to each other and must therefore coincide with *AB*.

34 *Theorem.* If a straight line is perpendicular to one of two parallels, it must also be perpendicular to the other.

Proof. Thus, if *EMB* (fig. 14), is a right angle, its equal *END* must also be a right angle

35. *Theorem.* Reciprocally, if two straight lines lying in the same plane, are perpendicular to a third they are parallel.

Proof. For the line, drawn through the point *M* parallel to *CD*, must be perpendicular to *EF*, and must therefore coincide with *AB*.

36. *Theorem.* If two straight lines, as *AB*, *CD* (fig. 15), are parallel to a third, *EF*, they are parallel to each other.

Proof. For, by the definition of parallel lines, they have the same direction with this third, and are therefore parallel

CHAPTER VI.

PERPENDICULAR AND OBLIQUE LINES.

37. *Theorem.* Only one perpendicular can be drawn from a point to a straight line. .

Proof. For, if two perpendiculars are erected in the same plane, at two different points, *M* and *P* (fig. 16) of the line *AB*, they are parallel, by § 35, and cannot meet at any point, as *C*.

38. *Theorem.* Two *oblique* lines, as *CE* and *CF* (fig. 17), drawn from the point *C* to the line *AB*, at equal distances *DE* and *DF* from the perpendicular *CD*, are equal.

Proof. For, if CDB be folded over upon CDA, DB will fall upon DA, because the right angles CDB and CDA are equal; the point F will fall upon E, because DF and DE are equal; and the straight lines CF and CE will coincide.

39. *Theorem.* A perpendicular measures the shortest distance of a point from a straight line.

Proof. Let the perpendicular CD (fig. 18) and the oblique line CF be drawn from the point C to the line AB. Produce CD to DE, making DE equal to DC, and join FE, we shall, by § 18, have

$$CE < FC + FE.$$

But

$$CE = 2\,CD,$$

and

$$FC + FE = 2\,FC,$$

for FC and FE are equal, because they are oblique lines drawn from the point F to the line CE at equal distances DC and DE from the perpendicular.

Therefore

$$2\,CD < 2\,FC,$$

or

$$CD < FC.$$

40. *Lemma.* The sum of two lines, as CA and CB (fig. 19), drawn to the extremities of the line AB, is greater than that of two other lines DA and DB, similarly drawn, but included by them.

Proof. Produce DA to E.

We have, by § 18,

$$AC + CE > AD + DE,$$

and

$$DE + BE > DB$$

The sum of these inequalities is

$$AC + CE + DE + BE > AD + DE + DB,$$

or, striking out the common term DE, and substituting for $CE + BE$, its equal BC,

$$AC + BC > AD + DB.$$

41. *Theorem.* Of two oblique lines, CF and CG (fig. 18), drawn unequally distant from the perpendicular, the more remote is the greater.

Proof. For, the figure being constructed as in § 39, and GE being joined, we have, by the preceding proposition,

$$GC + GE > FC + FE;$$

or, as in § 39,

$$2\ GC > 2\ FC,$$

and

$$GC > FC.$$

42. *Theorem.* If from the point C the middle of the straight line AB (fig. 20), a perpendicular EC be drawn: —

1. Any point in the perpendicular EC is equally distant from the two extremities of the line AB.

2. Any point without the perpendicular, as F, is at unequal distances from the same extremities A and B.

Proof. 1. The distances EA and EB are equal, since they are oblique lines drawn at equal distances CA and CB from the perpendicular AB.

2. The distance FA is greater than FB; for

$$FA = FE + EA$$
$$= FE + EB$$

while

$$FE + EB > FB.$$

CHAPTER VII.

SIDES AND ANGLES OF POLYGONS.

43. *Definitions.* A *plane figure* is a plane terminated on all sides by lines.

If the lines are straight, the space which they contain is called a *rectilineal figure*, or *polygon* (fig. 21), and the sum of the bounding lines is the *perimeter* of the polygon.

44. *Definitions.* The polygon of three sides is the most simple of these figures, and is called a *triangle*; that of four sides is called a *quadrilateral*; that of five sides, a *pentagon*; that of six, a *hexagon*, &c.

45. *Definitions.* A triangle is denominated *equilateral* (fig. 22), when the three sides are equal, *isosceles* (fig. 23), when two only of its sides are equal, and *scalene* (fig. 24), when no two of its sides are equal.

46. *Definitions.* A *right-triangle* is that which has a right angle. The side opposite to the right angle is called the *hypothenuse*. Thus ABC (fig. 25) is a triangle right-angled at A, and the side BC is the hypothenuse.

47. *Definitions.* Among quadrilateral figures, we distinguish

The *square* (fig. 26), which has its sides equal, and its angles right angles. (See § 73).

The *rectangle* (fig. 27), which has its angles right angles, without having its sides equal.

The *parallelogram* (fig. 28), which has its opposite sides parallel.

The *rhombus* or *lozenge* (fig. 29), which has its sides equal without having its angles right angles.

The *trapezoid* (fig. 30), which has two only of its sides parallel.

48. *Definition.* A *diagonal* is a line which joins the vertices of two angles not adjacent, as AC (fig. 30.)

49. *Definitions.* An *equilateral* polygon is one which has all its sides equal; an *equiangular* polygon is one which has all its angles equal.

50. *Definition.* Two polygons are *equilateral with respect to each other*, when they have their sides equal, each to each, and placed in the same order; that is, when, by proceeding round in the same direction, the first in the one is equal to the first in the other, the second in the one to the second in the other, and so on.

In a similar sense are to be understood two polygons *equiangular with respect to each other.*

The equal sides in the first case, and the equal angles in the second, are called *homologous.*

51. *Theorem.* Two triangles are equal, when two sides and the included angle of the one are respectively equal to two sides and the included angle of the other.

Proof. In the two triangles ABC, DEF, (fig. 31), let the angle A be equal to the angle D, and the sides AB, AC, respectively equal to DE, DF

Place the side *DE* upon its equal *AB*. *DF* will take the direction *AC*, because the angle *D* is equal to the angle *A*; the point *F* will fall upon *C*, because *DF* is equal to *AC*; and the lines *FE* and *BC* will coincide, since their extremities are the same points. The triangles will therefore coincide, and must be equal.

52. *Corollary.* Hence, when two sides and the included angle of one triangle are respectively equal to those of another, the other side and angles are also equal in the two triangles.

53. *Theorem.* Two triangles are equal, when a side and the two adjacent angles of one triangle are respectively equal to those of the other.

Proof. In the two triangles *ABC*, *DEF* (fig. 31), let the side *AB* be equal to the side *DE*, and the angles *A* and *B* respectively equal to *D* and *E*.

Place the side *DE* upon the side *AB*. The side *DF* will take the direction *AC*, because the angle *D* is equal to *A*; the side *EF* will take the direction *BC*, because the angle *E* is equal to *B*; and the point *F*, falling at once in each of the lines *AC* and *BC*, must fall upon their point of intersection *C*. The triangles will therefore coincide, and must be equal

54. *Corollary.* Hence, when a side and the two adjacent angles of one triangle are respectively equal to those of another, the other sides and angle are also equal in the two triangles.

55. *Theorem.* In an isosceles triangle the angles opposite the equal sides are equal.

Proof. In the isosceles triangle *ABC* (fig. 32), let the equal sides be *AB* and *BC*.

Let the line *BD* be drawn so as to bisect the angle *ABC*

Then the two triangles *ADB* and *DBC* will be equal, since they have two sides *AB*, *BD*, and the included angle *ABD*, respectively equal to the two sides *BC*, *BD*, and the included angle *DBC*; and the angle *A* will be equal to *C*.

56. *Corollary*. An equilateral triangle is also equiangular.

57. *Theorem*. The line *BD* (fig. 32), which bisects the angle *B*, at the vertex of an isosceles triangle, is perpendicular to the base, and bisects the base.

Proof. *a*. For, on account of the equality of the triangles *ABD* and *BCD*, *AD* must be equal to *DC*.

b. Moreover, the angles *BDA* and *BDC* are equal, and are therefore right angles by the very definition of the right angle in § 20.

58. *Theorem*. If, in a triangle, two angles are equal, the opposite sides are also equal, and the triangle is isosceles.

Proof. In the triangle *ABC* (fig. 32), let the angle *A* be equal to the angle *C*.

Invert the triangle, and place it in the position *BCA*; and, as the two triangles *ABC* and *CBA* have the side *AC* and the adjacent angles *A* and *C* of the one respectively equal to *CA* and the adjacent angles *C* and *A* of the other, their other sides must be equal, or *BC* must be equal to *BA*.

59. *Corollary*. An equiangular triangle is also equilateral

60. *Lemma.* Two different triangles cannot be formed on a given line AB (fig. 33), of which the sides, AD and DB, are respectively equal to CA and CB, and terminate at the same extremities of AB.

Proof. For, first, the vertex D of one triangle cannot fall within the other triangle ACB, as in fig. 19, because, by § 40, $AD + DB$ must in this case be less than $AC + CB$.

Secondly. If D falls without ACB, as in fig. 33, the triangles ACD and BCD are isosceles, since AC is equal to AD and BC is equal to BD.

Hence $$ACD = ADC,$$
and $$BCD = BDC;$$
but this is impossible; for of the first members of these equations $$ACD > BCD$$
while of the second members
$$ADC < BDC.$$

61. *Theorem.* When two triangles are equilateral with respect to each other, they must be equal, and must also be equiangular with respect to each other.

Proof. Let ABC and DEF (fig. 31) be the triangles, whose sides AB, BC, and AC are respectively equal to DE, EF, and DF.

If DE is placed upon AB, the point F must by the preceding proposition fall upon C, and the triangles must coincide.

62. *Theorem.* Of two sides of a triangle, that is the greater which is opposite the greater angle; and conversely, of two angles of a triangle, that is the greater which is opposite the greater side.

The greatest Side of a Triangle opposite the greater Angle.

Proof. 1. Suppose the angle $C > B$ (fig. 34). Draw CD so as to make the angle $BCD = B$.

Then will $$BD = CD$$
and $$AB = AD + DB = AD + DC$$
But $$AD + DC > AC$$
Hence $$AB > AC.$$

2. *Conversely.* Suppose $AB > AC$, the angle C must be greater than B, for if C were equal to or less than B, AB would by § 61 and the preceding demonstration, be equal to or less than AC.

63. *Theorem.* If two triangles have two sides of the one respectively equal to two sides of the other, and if the included angle of the first triangle is greater than the included angle of the second triangle, the third side of the first triangle, is also greater than the third side of the second triangle.

Proof. Let the first triangle be ABC (figs. 19 and 33), and the second ABD, which have the sides AB and AD, respectively equal to AB and AC, and the included angles $BAD < BAC$.

1. If the point D falls within the first triangle as in fig. 19, we have by § 40

$$AC + BC > AD + DB\,;$$

whence, substracting the equals AC and AD,

$$BC > BD.$$

2. If the point D falls upon the third side as at E, we have at once

$$BC > BE.$$

3. If the point D falls without the first triangle, as in fig. 33, we have in the isosceles triangle ACD,

$$ACD = ADC.$$

But $BDC > ADC$, while $ACD > BCD$;

whence $BDC > BCD$,

so that in the triangle BCD, by § 62, we have

$$BC > BD.$$

64. *Theorem.* Two right triangles are equal, when the hypothenuse and a side of the one are respectively equal to the hypothenuse and a side of the other.

Proof. Let ABC and DEF (fig. 35) be the right triangles, of which the hypothenuse AC is equal to DF, and AB equal to DE.

Place DE upon AB, EF will fall upon CB produced, since the right angles ABG and DEF are equal. An isosceles triangle CAG is thus formed, and AB being perpendicular to its base, divides it, by § 57, into the two equal triangles ABC and ABG.

65. *Theorem.* The sum of the three angles of any triangle is equal to two right angles.

Proof. Let ABC (fig. 36) be the given triangle. Produce AC to D, and draw CE parallel to AB.

The angles ABC and BCE, being alternate-internal angles, are equal, and BAC and ECD, being external-internal angles, are equal. Hence the sum of the three angles of the triangle is equal to $ACB + BCE + ECD$, or, by § 25, to two right angles.

66. *Corollary.* Two angles of a triangle being given, or only their sum, the third will be known by subtracting the sum of these angles from two right angles.

67. *Corollary.* If two angles of one triangle are respectively equal to two angles of another triangle, the third of the one is also equal to the third of the other.

and the two triangles are equiangular with respect to each other.

68. *Corollary.* In a triangle, there can only be one right angle, or one obtuse angle.

69. *Corollary.* In a right triangle, the sum of the acute angles is equal to a right angle.

70. *Corollary.* An equilateral triangle, being also equiangular, has each of its angles equal to a third of two right angles, or $\frac{2}{3}$ of one right angle.

71. *Corollary.* In any triangle ABC, if we produce the side AC toward D, the exterior angle BCD is equal to the sum of the two opposite interior angles A and B.

72. *Theorem.* The sum of all the interior angles of a polygon is equal to as many times two right angles as it has sides minus two.

Proof. Let $ABCDE$, &c. (fig. 37), be the given polygon.

Draw from either of the vertices, as A, the diagonals AC, AD, AE, &c.

The polygon will obviously be divided into as many triangles as it has sides minus two, and the sum of the angles of these triangles is the same as that of the angles of the polygon. But the sum of the angles of each triangle is, by § 65, equal to two right angles; and, consequently, the sum of all their angles is equal to as many times two right angles as there are triangles, that is, as there are sides to the polygon minus two.

73. *Corollary.* The sum of the angles of a quadrilateral is equal to two right angles multiplied by $4 - 2$;

which makes four right angles; therefore, if all the angles of a quadrilateral are equal, each of them will be a right angle, which justifies the definition of a square and rectangle of § 47.

74. *Corollary.* The sum of the angles of a pentagon is equal to two right angles multiplied by 5 — 2, which makes 6 right angles; therefore, when a pentagon is equiangular, each angle is the fifth of six right angles, or $\frac{6}{5}$ of one right angle.

75. *Corollary.* The sum of the angles of a hexagon is equal to 2 × (6 — 2), or 8 right angles; therefore, in an equiangular hexagon, each angle is the sixth of eight right angles, or $\frac{4}{3}$ of one right angle. The process may be easily extended to other polygons.

76. *Scholium.* If we would apply this proposition to polygons, which have any angles whose vertices are directed inward, as *CDE* (fig. 38), each of these angles is to be considered as greater than two right angles. But, in order to avoid confusion, we shall confine our selves in future to those polygons, which have angles directed outwards, and which may be called *convex* polygons. Every *convex* polygon is such, that a straight line, however drawn, cannot meet the perimeter in more than two points.

77. *Theorem.* The diagonal of a parallelogram divides it into two equal triangles.

Proof. Let *ABCD* (fig. 39) be the parallelogram and *AC* its diagonal.

The two triangles *ABC* and *ADC* are equal, since they have the side *AC* common, the angle *BAC* = *ACD*, by

§ 30, on account of the parallels AB and CD, and $BCA = CAD$, on account of the parallels BC and AD.

78. *Theorem.* The opposite sides of a parallelogram are equal, and the opposite angles are equal.

Proof. For the triangles ACB and ACD (fig. 39) being equal, their sides CB and AB are respectively equal to AD and DC; and the angle $ABC = ADC$. In the same way it might be proved that $BAD = BCD$.

79. *Corollary.* Two parallel lines comprehended between two other parallel lines are equal.

80. *Theorem.* If, in a quadrilateral $ABCD$ (fig. 39), the opposite sides are equal, namely, $AB = CD$, and $AD = BC$, the equal sides are parallel, and the figure is a parallelogram.

Proof. For the triangles ABC and ACD are equal, having their three sides respectively equal; and therefore $ACB = CAD$, whence BC is parallel to AD, by § 31; and $BAC = ACD$, whence AB is parallel to CD.

81. *Theorem.* If two opposite sides BC, AD (fig. 39) of a quadrilateral are equal and parallel, the two other sides are also equal and parallel, and the figure $ABCD$ is a parallelogram.

Proof. For the triangles ABC and ACD are equal, since they have the side AC common, the side $BC = AD$, and the included angle $BCA = CAD$, on account of the parallelism of BC and AD; and therefore AB and CD must be equal and parallel.

82. *Theorem.* Two parallel lines are throughout at the same distance from each other.

Proof. The two parallels AB and CD (fig. 40), being given, if through two points taken at pleasure we erect, upon AB, the two perpendiculars EG and FH, the straight lines EG, FH will, by § 34, be perpendicular to CD; and they are also parallel and equal to each other, by arts. 35 and 79.

83. *Theorem.* The two diagonals of a parallelogram mutually bisect each other.

Proof. For the triangles (fig. 41) ADO and BOC are equal, since the side $BC = AD$, and the angles $OCB = OAD$, and $OBC = ODA$, on account of the parallelism of BC and AD; therefore $AO = OC$ and $BO = OD$.

84. *Corollary.* In the case of the rhombus (fig. 42), the triangles AOB and AOD are equal, for the sides $AB = AD$, $BO = DO$, and AO is common; therefore the angles AOB and AOD are equal, and, as they are adjacent, each of them must, by definition, § 20, be a right angle, so that *the two diagonals of a rhombus bisect each other at right angles.*

CHAPTER VIII.

THE CIRCLE AND THE MEASURE OF ANGLES.

85. *Definitions.* The *circumference of a circle* is a curved *line*, all the points of which are equally distant from a point within, called the *centre*.

The *circle* is the *space* terminated by this curved line.

86. *Definitions.* The *radius* of a circle is the straight

line, as AB, AC, AD (fig. 43), drawn from the centre to the circumference.

The *diameter* of a circle is the straight line, as BD, drawn through the centre, and terminated each way by the circumference.

87. *Corollary.* Hence, all the radii of a circle are equal, and all its diameters are also equal, and double of the radius.

88. *Theorem.* Every diameter, as BD (fig. 43), bisects the circle and its circumference.

Proof. For if the figure BCD be folded over upon the part BED, they must coincide; otherwise there would be points in the one or the other unequally distant from the centre.

89. *Definition.* A *semicircumference* is one half of the circumference, and a *semicircle* is one half of the circle itself.

90. *Definition.* An *arc* of a circle is any portion of its circumference, as BFE.

The *chord* of an arc is the straight line, as BE, which joins its extremities.

The *segment* of a circle, is a part of a circle comprehended between an arc and its chord, as EFB.

91. *Theorem.* Every chord is less than the diameter.

Proof. Thus BE (fig. 43) is less than DB For, joining AE, we have $BD = BA + AE$, but $BE < BA + AE$, therefore $BE < BD$.

92. *Definition.* A straight line is said to be ***inscribed in a circle***, when its extremities are in the circumference of the circle

93. *Corollary.* Hence the greatest straight line which can be inscribed in a circle is equal to its diameter.

94. *Theorem.* A straight line cannot meet the circumference of a circle in more than two points.

Proof. For, by §§ 38 and 41, only two equal straight lines can be drawn from the same point to the same straight line; whereas, if a straight line could meet the circumference *ABD* (fig. 45) in the three points, *ABD*, three equal straight lines *CA*, *CB*, *CD*, would be drawn from the point *C* to this line.

95. *Theorem.* In the same circle, or in equal circles, equal angles *ACB*, *DCE* (fig. 44), which have their vertices at the centre, intercept upon the circumference equal arcs *AB*, *DE*.

Proof. Since the angles *DCE* and *ACB* are equal, one of them may be placed upon the other; and since their sides are equal, the point *D* will fall upon *A*, and the point *E* upon *B*. The arcs *AB* and *DE* must therefore coincide, or else there would be points in one or the other unequally distant from the centre.

96. *Theorem.* Reciprocally if the arcs *AB*, *DE* (fig. 44) are equal, the angles *ACB* and *DCE* must be equal.

Proof. For if the line *CE* be drawn, so as to make an angle *DCE* equal to *ACB*, it must pass through the extremity *E* of the arc *DE*, which is equal to *AB*.

97. *Theorem.* Two angles, as *ACB*, *ACD* (fig 45), are to each other as the arcs *AB*, *AD* intercepted

between their sides, and described from their vertices as centres, with equal radii.

Proof. Suppose the less angle placed in the greater, and suppose the angles to be to each other, for example, as 7 to 4; or, which amounts to the same, suppose the angle *AC a*, which is their common measure, to be contained 7 times in *ACD*, and 4 times in *ACB*; so that the angle *ACD* may be divided into the 7 equal angles *AC a*, *a C b*, *b C c*, &c., while the angle *ACB* is divided into the 4 equal angles *AC a*, &c.

The arcs *AB* and *AD* are, at the same time, divided into the equal parts *A a*, *ab*, *bc*, &c., of which *AD* contains 7 and *AB* 4; and therefore these arcs must be to each other as 7 to 4, that is, as the angles *ACD* and *ACB*.

98. *Scholium.* The preceding demonstration does not strictly include the case in which the two angles are *incommensurable*, that is, in which they have no common divisor. The divisor *AC a*, instead of being contained an exact number of times in the given angles *ACB*, *ACD*, is, in this case, contained in one or each of them a certain number of times plus a remainder less than the divisor. So that if these remainders be neglected, the angle *AC a* will be a common divisor of the given angles.

Now the angle *AC a* may be taken as small as we please; and therefore the remainders, which are neglected, may be as small as we please; less, then, than any assignable quantity, less than any *conceivable* quantity, that is, less than any *possible* quantity within the limits of human knowledge. Such quantities can, undoubtedly, be neglected, *without any error*; and the above demonstration is thus extended to the case of incommensurable angles

99. The principle, involved in the reasoning just given, is general in its application; and may be stated as follows, using the term *infinitely small quantity* to denote a quantity which may be taken at pleasure, as small as we please, so that it may be supposed equal to nothing whenever we please.

Axiom. Infinitely small quantities may be neglected.

100. *Corollary.* Since the angle at the centre of a circle is proportional to the arc included between its sides, either of these quantities may be assumed as the *measure* of the other; and we shall, accordingly, adopt, *as the measure of the angle, the arc described from its vertex as a centre and included between its sides.*

But when different angles are compared with each other, the arcs, which measure them, must be described with equal radii.

101. *Definitions.* In order to compare together different arcs and angles, every circumference of a circle may be supposed to be divided into 360 equal arcs called *degrees*, and marked thus ($^\circ$). For instance, 60° is read 60 degrees.

Each degree may be divided into 60 equal parts called *minutes*, and marked thus ($'$).

Each minute may be divided into 60 equal parts called *seconds*, and marked thus ($''$).

When extreme minuteness is required, the division is sometimes extended to *thirds* and *fourths*, &c., marked thus ($'''$), ($''''$), &c.

A *quadrant* is a fourth part of a circumference, and contains 90°. This is called the *sexagesimal* division of the circle; another which is called the *centesimal* di-

vision has been introduced by the French geometers. They divide the quadrant into 100 degrees, the degree into 100 minutes, &c; so that by this method of division, the whole notation is decimal.

102. *Scholium.* As all circumferences, whether great or small, are divided into the same number of parts, it follows that a degree which is thus made the *unit* of arcs, is not a fixed value, but varies for every different circle. It merely expresses the ratio of an arc, namely, $\frac{1}{360}$ to the whole circumference of which it is a part, and *not* to any other.

103. *Corollary.* The angle may be designated by the degrees and minutes of the arc which measures it; thus the angle which is measured by the arc of 17° 28′ may be called the angle of 17° 28′.

104. *Corollary.* The *right angle* is then an angle of 90°, and is measured by the quadrant.

105. *Corollary.* The angle which is measured by the arc of one degree, that is, the angle of 1° is then $\frac{1}{90}$ of a right angle, and has a fixed value, altogether independent, in its magnitude, of the radius of the arc by which it is measured.

The same is the case with an angle of any other value, so that the arcs AP, $A'D'$, $A''D''$, &c. (fig. 46), of the same number of degrees, all measure the same angle C, the vertex of which is at their common centre.

106. *Definitions.* An *inscribed angle* is one, whose vertex is in the circumference of a circle, and which is formed by two chords, as BAC (fig. 47).

An *inscribed triangle* is a triangle whose three angles have their vertices in the circumference of the circle.

And, in general, an ***inscribed figure*** is one, all whose angles have their vertices in the circumference of the circle. In this case the circle is said to be ***circumscribed*** about the figure.

107. The inscribed angle BAC (figs. 47, 48, 49) has for its measure the half of the arc BC comprehended between its sides.

Proof. 1. If one of the sides is a diameter, as AC (fig. 47), O being the centre of the circle,

Join BO. Then the triangle AOB is isosceles, for the radii AO, BO are equal. Therefore the angles OAB and OBA are equal, and the exterior angle BOC being equal to their sum, by § 71, is equal to the double of either of them, as BAC. BAC is, therefore, half of BOC and has half its measure, or half of BC.

2. If the centre O falls within the angle, as in (fig. 48,)

Draw the diameter AOD; and, by the above, BAD has for its measure half of BD, and DAC half of DC; so that $BAD + DAC$ or BAC has for its measure half of $BD + DC$, or half of BC.

3. If the centre O falls without the angle, as in (fig. 49,)

Draw the diameter AOD; and $BAD - DAC$, or BAC has for its measure half of $BD - DC$, or half of BC.

108. *Corollary.* All the angles BAC, BDC (fig. 50), &c., inscribed in the same segment are equal.

Proof. For they have each for their measure the half of the same arc BEC.

109. *Corollary.* Every angle BAD (fig. 51) inscribed in a semicircle is a right angle.

Arcs and Chords.

Proof. For it has for its measure the half of the semicircumference BED, or a quadrant.

110. *Corollary.* Every angle BAC (fig. 50) inscribed in a segment greater than a semicircle is an acute angle, for it has for its measure the half of an arc BEC less than a semicircumference.

111. *Corollary.* Every angle BEC inscribed in a segment less than a semicircle is an obtuse angle; for it has for its measure the half of an arc greater than a semicircumference.

112. *Theorem.* In the same circle, or in equal circles, equal arcs are subtended by equal chords.

Proof. Let the arc AB (fig. 52) be equal to the arc BC.

Join AC; and, in the triangle ABC, the angles A and C are equal, for they are measured by the halves of the equal arcs BC and AB. The triangle ABC is therefore isosceles, by § 58, and the chords AB and BC are equal.

113. *Theorem.* Conversely, in the same circle, or in equal circles, equal chords subtend equal arcs.

Proof. Let the chord AB (fig. 52) be equal to the chord BC.

Join AC; and in the isosceles triangle ABC the angles A and C must be equal, by § 55, and also the arcs AB and BC, which are double their measures.

114. *Theorem.* In the same circle, or in equal circles, if the sum of two arcs be less than a circumference, the greater arc is subtended by the greater chord; and, conversely, the greater chord is subtended by the greater arc.

Proof. *a.* Let the arc *BC* (fig. 53) be greater than the arc *AB*.

Join *AC*; and the angle *BAC*, being measured by half the arc *BC*, is greater than *BCA*, which is measured by half of *AB*; and therefore, by § 62, the chord *BC* is greater than *AB*.

b. Conversely. Suppose the chord $BC > AB$.

Join *AC*; and, by § 62, $BAC > BCA$, and, therefore, the arc *BC* double the measure of *BAC* is greater than the arc *AB* double the measure of *BCA*.

115. *Corollary.* If the sum of the two arcs is greater than a circumference, the greater arc is subtended by the less chord, and the less arc by the greater chord.

Proof. Suppose the arc $BCNA > BANC$ (fig. 53).

Take *ANC* from each, and we have the arc $BC > BA$, and consequently, by the preceding proposition, the chord *BC* of the less arc *BANC* is greater than the chord *BA* of the greater arc *BCNA*.

116. *Theorem.* The radius *CG* (fig. 54), perpendicular to a chord *AB*, bisects this chord and the arc subtended by it.

Proof. *a.* The radii *CA* and *CB* are equal oblique lines drawn to the chord *AB*. They are, therefore, by § 38, at equal distances from the perpendicular, or $AD = DB$.

b. Since the line *GC* is a perpendicular erected at the middle of the straight line *AB*, any point of it, as *G*, is, by § 42, at equal distances from its extremities, that is, the chords *AG* and *GB* are equal; and therefore, by § 113, the arcs *AG* and *GB* are equal.

117. *Corollary.* The perpendicular erected upon the middle of a chord passes through the centre, and also through the middle of the arc subtended by the chord.

118. *Definitions.* A *secant* is a line which meets the circumference of a circle in two points, as *AB* (fig. 55).

A *tangent* is a line, which has only one point in common with the circumference, as *CD.*

The common point *M* is called the point of *contact*

Also two circumferences are tangents to each other (figs. 56 and 57), when they have only one point common.

A polygon is said to be *circumscribed* about a circle, when all its sides are tangents to the circumference; and in this case the circle is said to be *inscribed* in the polygon.

119. *Theorem.* The direction of the tangent is the same as that of the circumference at the point of contact.

Proof. Draw through the point *M* (fig. 55) the secant *ME* and the tangent *MD.*

If the secant *ME* is turned around the point *M* so as to diminish the angle *EMD,* the secant *ME* will approach the tangent *MD,* and the point *E* will approach the point *M.* When *ME* is turned so far as to pass through the point *P* next to *M,* the angle *DME* will be infinitely small, since *P* is at an infinitely small distance from *M*; and the line *ME* will approach infinitely near the tangent *MD,* that is, it will, by § 99, coincide with this tangent, which has therefore, by § 11, the same direction with the circumference at *M*

120. *Theorem.* The tangent to a circle is perpendicular to the radius drawn to the point of contact.

Proof. The radius $OM = ON$ (fig. 58) is shorter than any other line, as OP, which can be drawn from the point O to the tangent MP; it is therefore, by § 39, perpendicular to this tangent.

121. The angle BAC (fig. 59), formed by a tangent and a chord, has for its measure half the arc BMA comprehended between its sides.

Proof *a.* Draw the diameter AD, and we have

$$BAC = DAC - DAB.$$

But DAC, being a right angle, has for its measure half of a semicircumference, as ABD; also BAD has, by § 107, for its measure half of the arc BD. The measure of BAC is therefore

$$\tfrac{1}{2}\,(ABD - BD) = \tfrac{1}{2}\,AMB.$$

b. In the same way, it may be shown that BAE has for its measure half the arc BDA.

122. *Theorem.* The angle BAC, formed by two secants (fig. 60), two tangents (fig. 62), or a tangent and a secant (fig. 61), and which has its vertex without the circumference of the circle, has for its measure half the concave arc BMC intercepted between its sides, minus half the convex arc DNE.

Proof. Join BE; and as BEC is an exterior angle of the triangle ABE, we have, by § 71

$$BEC = ABE + BAC,$$

whence

$$BAC = BEC - ABE.$$

But the measure of BEC is half of BMC, and that of ABE is half of DNE; therefore the measure of BAC is

$$\tfrac{1}{2} BMC - \tfrac{1}{2} DNE.$$

Scholium. In applying the preceding demonstration to (figs. 61 and 62), the letters B and D must denote the same point; and in (fig. 62) the letters C and E must also denote the same point.

123. *Theorem.* The angle BAC (fig. 63), formed by two chords, and which has its vertex between the centre and the circumference, has for its measure half the arc BC contained between its sides plus half the arc DE contained between its sides produced.

Proof. Join BE; and, as BAC is an exterior angle of the triangle ABE, we have, by § 71,

$$BAC = BEA + ABE.$$

But the measure of BEA is, by § 107, half of BC; and that of ABE is half of DE; therefore the measure of BAC is

$$\tfrac{1}{2} BC + \tfrac{1}{2} DE.$$

124. *Theorem.* Two parallels AB and DC (figs. 64, 65, 66), intercept upon the circumference equal arcs AD, BC.

Proof. Join BD. The alternate-internal angles ABD and BDC are equal, by § 30; and therefore, the arcs AD and BC, the double of their measures, are equal.

Scholium. In applying this demonstration to figs. (65 and 66), the letters A and B must denote the same point; and in (fig. 66) the letters D and C must also denote the same point.

125. *Corollary.* The arcs AD and BC (fig. 66) being equal must be semicircumferences, and the chord BC must be a diameter

126. *Theorem*. When the circumferences of two circles cut each other, the line *AB* (fig. 67), which joins their centres, is perpendicular to the middle of the line *CD*, which joins their points of intersection.

Proof. For if a perpendicular be erected upon the middle of the chord *CD*, it must, by § 117, pass through the centres *A* and *B* of both the circles of which *CD* is a chord.

127. *Theorem*. When two circumferences are tangents to each other, their centres and point of contact are in the same straight line perpendicular to their common tangent at the point of contact.

Proof. *a*. If the centres of two circumferences which cut each other (fig. 67) are removed from each other, until the points *C* and *D* of intersection approach infinitely near to each other, the circles will become tangent, as in (fig. 56), the chord *CD* of (fig. 67) will become the tangent *CD* of (fig. 56); and as both the radii *AM* and *MB* are perpendicular to their common tangent, these radii must be in the same straight line.

b. In the same way, the centres of the circles (fig. 67) may be brought near to each other until the circles are tangents, as in (fig. 57), and the same reasoning may be here applied to prove that the line *ABM*, perpendicular to the common tangent at *M*, passes through both the centres *A* and *B*

CHAPTER IX.

PROBLEMS RELATING TO THE FIRST EIGHT CHAPTERS.

128. *Problem.* To find the position of a point in a plane, having given its distances from two known points in that plane.

Solution. Let the known points be A and B (fig. 68). From the point A as a centre, with a radius equal to the distance of the required point from A, describe an arc. Also, from the point B as a centre, with a radius equal to the distance of the required point from B, describe an arc cutting the former arc ; and the point of intersection C is the required point.

Scholium. By the same process, another point D may also be found which is at the given distances from A and B, and either of these points therefore satisfies the conditions of the problem.

129. *Corollary.* If both the radii were taken of equal magnitudes, the points C and D thus found would be at equal distances from A and B.

130. *Scholium.* The problem is impossible, when the distance between the known points is greater than the sum of the given distances or less than their difference.

131. *Scholium.* If the required point is to be at equal distances from the known point, its distance from either of them must be greater than half the distance between the known points.

To Bisect a Line; to Erect a Perpendicular.

132. *Problem.* To divide a given straight line *AB* (fig. 69) into two equal parts; that is, to *bisect* it.

Solution. Find by § 129, a point *C* at equal distances from the extremities *A* and *B* of the given line. Find also another point *D*, either above or below the line, at equal distances from *A* and *B*. Through *C* and *D* draw the line *CD*, which bisects *AB* at the point *E*.

Proof. For the perpendicular, erected at *E* to the line *AB*, must, by § 42, pass through the points *C* and *D*, and must therefore, by § 16, coincide with the line *CD*.

133. *Problem.* At a given point *A* (fig. 70), in the line *BC*, to erect a perpendicular to this line.

Solution. Take the points *B* and *C* at equal distances from *A*; and find a point *D* equally distant from *B* and *C*. Join *AD*, and it is the perpendicular required.

Proof. For the point *D* must, by § 42, be a point of the perpendicular erected at *A*.

134. *Problem.* From a given point *A* (fig. 71), without a straight line *BC*, to let fall a perpendicular upon this line.

Solution. From *A* as a centre, with a radius sufficiently great, describe an arc cutting the line *BC* in two points *B* and *C*; find a point *D* equally distant from *B* and *C*, and the line *ADE* is the perpendicular required.

Proof. For the points *A* and *D*, being equally distant from *B* and *C*, must, by § 42, be in this perpendicular.

135. *Problem.* To make an arc equal to a given arc *AB* (fig. 72), the centre of which is at the given point *C*.

Solution. Draw the chord *AB*. From any point *D* as a centre, with a radius equal to the given radius *CA*,

describe the indefinite arc FH. From F as a centre, with a radius equal to the chord AB, describe an arc cutting the arc FH in H, and we have the arc $FH = AB$.

Proof. For as the chord $AB =$ the chord FH, it follows, from § 112, that the arc $AB =$ the arc FH.

136. *Problem.* At a given point A (fig. 73), in the line AB, to make an angle equal to a given angle K

Solution. From the vertex K, as a centre, with any radius describe an arc IL meeting the sides of the angle; and from the point A as a centre, by the preceding problem, make an arc BC equal to IL. Draw AC, and we have $A = K$.

Proof. For the angles A and K being, by §100, measured by the equal arcs BC and IL, are equal.

137. *Problem.* To bisect a given arc AB (fig. 74).

Solution. Find a point D at equal distances from A and B. Through the point D and the centre C draw the line CD, which bisects the arc AB at E.

Proof. Draw the chord AB. Since the points D and C are at equal distances from A and B, the line DC is, by § 132, perpendicular to the middle of the chord AB, and therefore by § 117, it passes through the middle E of the arc AB.

138. *Problem.* To bisect a given angle A (fig. 75).

Solution. From A as a centre, with any radius, describe an arc BC, and, by the preceding problem, draw the line AE to bisect the arc BC, and it also bisects the angle A.

Proof. The angles BAE and EAC are equal, for they are measured by the equal arcs BE and EC

To construct a Triangle.

139. ***Problem.*** Through a given point A (fig. 76), to draw a straight line parallel to a given straight line BC.

Solution. Join EA, and, by the preceding problem, draw AD, making the angle $EAD = AEF$, and AD is parallel to BC, by § 31.

140. ***Problem.*** Two angles of a triangle being given, to find the third.

Solution. Draw the line ABC (fig. 77). At any point B draw the line BD, to make the angle DBC equal to one of the given angles, and draw BE, to make EBD equal to the other given angle, and ABE is the required angle.

Proof. For these three angles are, by § 25, together equal two right angles.

141. ***Problem.*** Two sides of a triangle and their included angle being given, to construct the triangle.

Solution. Make the angle A (fig. 78) equal to the given angle, take AB and AC equal to the given sides, join BC, and ABC is the triangle required.

142. ***Problem.*** One side and two angles of a triangle being given, to construct the triangle.

Solution. If both the angles adjacent to the given side are not given, the third angle can be found by § 140.

Then draw AB (fig. 78) equal to the given side, and draw AC and BC, making the angles A and B equal to the angles adjacent to the given side, and ABC is the triangle required.

143. ***Problem.*** The three sides of a triangle being given, to construct the triangle.

Solution. Draw AB (fig. 78) equal to one of the given

sides, and, by § 128, find the point *C* at the given distances *AC* and *BC* from the point *C*, join *AC* and *BC*, and *ABC* is the triangle required.

144. *Scholium.* The problem is impossible, when one of the given sides is greater then the sum of the other two.

145. *Problem.* To construct a right triangle, when a leg and the hypothenuse are given.

Solution. Draw *AB* (fig. 79) equal to the given leg. At *A* erect the perpendicular *AC*, from *B* as a centre, with a radius equal to the given hypothenuse, describe an arc cutting *AC* at *C*. Join *BC*, and *ABC* is the triangle required.

146. *Problem.* The adjacent sides of a parallelogram and their included angle being given, to construct the parallelogram.

Solution. Make the angle *A* (fig. 80) equal to the given angle, take *AB* and *AC* equal to the given sides, find the point *D*, by § 128, at a distance from *B* equal to *AC*, and at a distance from *C* equal to *AB*. Join *BD* and *DC*, and *ABCD* is, by § 80, the parallelogram required.

147. *Corollary.* If the given angle is a right angle, the figure is a rectangle; and, if the adjacent sides are also equal, the figure is a square.

148. *Problem.* To find the centre of a given circle or of a given arc.

Solution. Take at pleasure three points *A*, *B*, *C* (fig. 81) on the given circumference or arc; join the chords *AB* and *BC*, and bisect them by the perpendiculars *DE* and *FG*; the point *O* in which these perpendiculars meet is the centre required.

Proof. For, by § 117, the perpendicular DE and FG must both pass through the centre, which must therefore be at their point of meeting.

149. *Scholium.* By the same construction a circle may be found, the circumference of which passes through three given points not in the same straight line, or in which a given triangle is inscribed.

150. *Problem.* Through a given point, to draw a tangent to a given circle.

Solution. a. If the given point A (fig. 82) is in the circumference, draw the radius CA, and through A draw AD perpendicular to CA, and AD is, by § 120, the tangent required.

b. If the given point A (fig. 83) is without the circle, join it to the centre by the line AC; upon AC as a diameter describe the circumference $AMCN$, cutting the given circumference in M and N; join AM and AN, and they are the tangents required.

Proof. For the angles AMC and ANC are right angles, because they are inscribed in semicircles, and therefore AM and AN are perpendicular to the radii MC and NC at their extremities, and are, consequently, tangents, by § 120.

151. *Corollary.* The two tangents AM and AN are equal; for the right triangles AMC and ANC are equal, by § 64, since they have the hypothenuse AC common, and the leg MC equal to the leg NC, and, therefore, the other legs AM and AN are equal.

152. *Problem.* To inscribe a circle in a given triangle ABC (fig. 84).

Solution. Bisect the angles A and B by the lines AO

and BO, and their point of intersection O is the centre of the required circle, and the perpendicular OD let fall from O upon the side AC is its radius.

Proof. The perpendiculars OD, OE, and OF let fall from O upon the sides of the triangle are equal to each other. For in the right triangles OAD and OAE the hypothenuse OA is common; the angle $OAD = OAE$ by construction; and the third angle $AOD = AOE$, by § 67; the triangles are, therefore, equal, by § 53; and OD is equal to OE. In the same way it may be proved that

$$OF = OD = OE.$$

Hence the circumference DFE passes through the points D, F, E, and the sides are tangents to it, by § 120.

153. *Corollary.* The three lines AO, BO, and CO, which bisect the three angles of a triangle, meet at the same point.

154. ***Problem.*** Upon a given straight line AB (figs. 85 and 86), to describe a segment capable of containing a given angle, that is, a segment such that each of the angles inscribed in it is equal to a given angle.

Solution. Draw BF, making the angle ABF equal to the given angle. Draw BO perpendicular to BF, and OC perpendicular to the middle of AB. From O, the point of intersection of OB and OC, with a radius $OB = OA$, describe the circumference $BMAN$, and BMA is the segment required.

Proof. Since BF is perpendicular to BO, it is a tangent to the circle, and therefore the angles AMB and ABF are equal, since they are each, by § 107 and 121, measured by half the arc ANB.

155. *Scholium.* If the given angle were a right angle,

the segment sought would be a semicircle described upon the diameter AB.

156. *Problem.* To find a common measure of two given straight lines, AB, CD (fig. 87), in order to express their ratio in numbers.

Solution. *a.* The method of finding the common divisor is the same as that given in arithmetic for two numbers. Apply the smaller CD to the greater AB, as many times as it will admit of; for example, twice with a remainder BE.

Apply the remainder BE to the line CD, as many times as it will admit of; twice, for example, with a remainder DF.

Apply the second remainder DF to the first BE, as many times as it will admit of; once, for example, with a remainder BG.

Apply the third remainder BG to the second DF, as many times as it will admit of.

Proceed thus till a remainder arises, which is exactly contained a certain number of times in the preceding.

This last remainder is a common measure of the two proposed lines; and, by regarding it as unity, the values of the preceding remainders are easily found, and, at length, those of the proposed lines from which their ratio in numbers is deduced.

If, for example, we find that GB is contained exactly three times in FD, GB is a common measure of the two proposed lines.

b. Let $$GB = 1;$$
and we have

$$\begin{aligned} FD &= 3\ GB = 3, \\ EB &= 1.\,FD + GB = 3 + 1 = 4, \\ CD &= 2.\,EB + FD = 8 + 3 = 11, \\ AB &= 2\ CD + EB = 22 + 4 = 26; \end{aligned}$$

consequently, the ratio of the lines AB, CD is as 26 to 11; that is, AB is $\frac{26}{11}$ of CD, and CD is $\frac{11}{26}$ of AB.

157. *Corollary.* By a like process, may be found the ratio of any two quantities, which can be successively applied to each other, like straight lines, as, for instance, two arcs or two angles.

CHAPTER X.

PROPORTIONAL LINES.

158. *Theorem.* If lines $a\,a'$, $b\,b'$, $c\,c'$, &c. (fig. 88), are drawn through two sides AB, AC of a triangle ABC, parallel to the third side BC, so as to divide one of these sides AB into equal parts $A\,a$, $a\,b$, &c., the other side AC is also divided into equal parts $A\,a'$, $a'b'$, &c.

Proof. Through the points a', b', c', &c. draw the lines $a'\,m$, $b'\,n$, $c'o$, &c. parallel to AB.

The triangles $A\,a\,a'$, $a'\,m\,b'$, $b'\,n'\,c$, &c. are equal, by § 53; for the sides $a'\,m$, $b'\,n$, $c'\,o$, &c. are, by § 79, respectively equal to $a\,b$, $b\,c$, $c\,d$, &c., and are therefore equal to each other and to $A\,a$; moreover, the angles $A\,a\,a'$, $m\,a'\,b'$, $n\,b'\,c'$, &c. are equal, by § 29, and likewise the angles $A\,a\,a'$, $a'\,m\,b'$, $b'\,n\,c'$, &c. Consequently, the sides $A\,a'$, $a'\,b'$, $b'\,c'$, &c. are equal.

159. *Problem.* To divide a given straight line AB (fig. 89) into any number of equal parts.

Solution. Suppose the number of parts is, for example, six. Draw the indefinite line AO; take AC of any convenient length, apply it six times to AO. Join B and the last point of division D by the line BD, draw CE parallel to DB, and AE, being applied six times to AB, divides it into six equal parts.

Proof. For if, through points of division of AD, lines are drawn parallel to DB, they must, by the preceding theorem, divide AB into six equal parts, of which AE is one.

160. *Theorem.* If a line DE (fig. 90) is drawn through two sides AB, AC of a triangle ABC, parallel to the third side BC, it divides those two sides proportionally, so that we have

$$AD : AB = AE : AC.$$

Proof. a. Suppose, for example, the ratio of $AD : AB$ to be as 4 to 7. AB may then be divided into 7 equal parts $A\,a$, $a\,b$, $b\,c$, &c., of which AD contains 4; and if lines $a\,a'$, $b\,b'$, $c\,c'$, &c. are drawn parallel to BC, AC is divided into 7 equal parts $A\,a'$, $a'\,b'$, $b'\,c'$, &c., of which AE contains 4. The ratio of AE to AC is, therefore, 4 to 7, the same as that of $AD : AB$.

161. *Scholium. b.* The case in which AD and AB are incommensurable, is included in this demonstration by the reasoning of § 98.

162. *Corollary.* In the same way

$$AD : BD = AE : EC.$$

and

$$BD : AB = EC : AC.$$

163. *Theorem. Conversely,* if a line DE (fig. 90)

Division of a Line into Parts proportional to given Lines.

is drawn so as to divide two sides AB, AC of a triangle proportionally, this line is parallel to the third side BC.

Proof For the line, which is drawn through the point D parallel to BC must, by the preceding proposition, pass through the point E, so as to divide the side AC proportionally to AB, and must therefore coincide with the proposed line DE.

164. ***Problem.*** To divide a given straight line AB (fig. 91) into two parts, which shall be in a given ratio, as in that of the two lines m to n.

Solution. Draw the indefinite line AO. Take $AC = m$ and $CD = n$. Join DB, through C draw CE parallel to DB; and E is the point of division required.

Proof. For, by § 161,

$$AE : EB = AC : CD = m : n.$$

165. ***Problem.*** To divide a given line AB (fig. 92) into parts proportional to any given lines, as m, n, o, &c.

Solution. Draw the indefinite line AO. Take

$$AC = m, \qquad CD = n, \qquad DE = o, \text{ \&c.}$$

Join B to the last point E, and draw CC', DD', &c. parallel to BE. C', D', &c. are the required points of division.

Proof. For, if AE is divided into parts equal each of them to the greatest common divisor of m, n, o, &c., and if, through the points of division, lines are drawn parallel to BE; it appears, from inspection, as in § 160, that

$$AC' : C'D' = AC : CD = m : n.$$

and that

$$C'D' : D'B = CD : DE = n : o;$$

or, as they may be written for brevity,

$$AC' : C'D' : D'B = m : n : o.$$

165. *Problem.* To find a line, to which a given line AB (fig. 93) has a given ratio, as that of the lines m to n; in other words, to find the fourth proportional to the three lines m, n, and AB.

Solution. Draw the indefinite line AB, take

$$AC = m, \qquad AD = n.$$

Join CB, draw DE parallel to BC, and AE is the required line.

Proof. For, by § 160,

$$AB : AE = AC : AD = m : n$$

166. *Corollary.* By making n equal to AB in the preceding solution, we find a third proportional to the two lines m and AB.

167. *Problem.* To divide one side BC (fig. 94), of a triangle ABC into two parts proportional to the other two sides.

Solution. Draw the line AD to bisect the angle BAC, and D is the required point of division, that is,

$$BD : DC = AB : AC.$$

Proof. Produce BA to E, making AE equal to AC. Join CE.

Then the angles ACE and AEC are equal, by § 55; and the exterior angle CAB of the triangle ACE is equal to $ACE + AEC$, or to $2\ CEA$, and, as DAB is half of BAC, we have

$$DAB = \tfrac{1}{2}(2\ CEA) = CEA,$$

and, therefore, by § 31, AD is parallel to CE, and, by § 161,

$$BD : DC = BA : AE,$$

or since $AE = AC$,

$$BD : DC = BA : AC.$$

168. *Problem.* Through a given point P (fig. 95) in a given angle A, to draw a line so that the parts intercepted between the point and the sides of the angle may be in a given ratio.

Solution. Draw PD parallel to AB. Take DC in the same ratio to AD as the parts of the required line. Through C and P draw CPE, and this is the required line.

Proof. For, by § 161,

$$CP : PE = CD : DA.$$

169. *Corollary.* When DC is taken equal to AD, PC is equal to PE.

CHAPTER XI.

SIMILAR POLYGONS.

170. *Definitions.* Two polygons are *similar*, which are equiangular with respect to each other, and have their *homologous sides* proportional.

In different circles, *similar arcs* are such as correspond to equal angles at the centre Thus the arcs AP, AD', &c. (fig. 46) are similar

171. *Definitions.* The *altitude of a parallelogram* is the perpendicular, which measures the distance between its opposite sides considered as bases.

The *altitude of a triangle* is the perpendicular, as AD (fig. 96), which measures the distance of any one of its vertices, as A, from the opposite side BC taken as a base.

The *altitude of a trapezoid* is the perpendicular, as EF (fig. 97), drawn between its two parallel sides.

172. *Theorem.* Two triangles ABC, DEF (fig. 98), which are equiangular with respect to each other, are similar.

Proof. Place the angle D upon its equal A; E must fall upon E', and F upon F'; and $F'E'$ is parallel to BC, because the angles $AE'F'$ and ACB are equal. Hence, by § 160,

$$AE' : AC = AF' : AB,$$

that is,

$$DE : AC = DF : AB.$$

In the same way, it may be proved that

$$DE : AC = EF : BC = DF : AB.$$

173. *Corollary.* Hence, and from § 67, it follows that two triangles are similar, when they have two angles of the one respectively equal to two angles of the other.

174. *Corollary.* Two right triangles are similar, when they have an acute angle of the one equal to an acute angle of the other.

175. *Theorem.* Two triangles are similar, when

they have the sides of the one respectively parallel to those of the other.

Proof. For, in this case, the angles are equal by § 29.

176. *Corollary.* The parallel sides are homologous.

177. *Theorem.* Two triangles are similar, when the sides of the one are equally inclined to those of the other, each to each, as ABC, DEF (fig. 99).

Proof. For if one of the triangles is turned around, by a quantity equal to the angle made by the sides of the one with those of the other, the sides of the two triangles become respectively parallel, and they are, therefore, by § 175, equiangular and similar.

178. *Corollary.* Two triangles are similar, when the sides of the one are respectively perpendicular to those of the other, and the perpendicular sides are homologous.

179. *Theorem.* Two triangles ABC, DEF (fig. 98) are similar, if they have an angle A of the one equal to an angle D of the other, and the sides including these angles proportional, that is,

$$AB : DF = AC : DE.$$

Proof. Place the angle D upon A; E falls upon E', and F upon F'; and $E'F'$ is parallel to BC, by § 162, because

$$AB : AF' = AC : AE'.$$

Hence, by § 30, the angle $C = AE'F' = E$,

and $B = AF'E' = F$;

that is, the triangles ABC and DEF are equiangular, and, by § 172, similar.

180. *Theorem.* Two triangles ABC, DEF (fig. 98) are similar, if they have their homologous sides proportional, that is,

$$AB : DF = AC : DE = BC : EF.$$

Proof. Take $AE' = DE$, and draw $E'F'$ parallel to BC. The triangles $AE'F'$ and ABC are similar, by § 175, and we are to prove that $AE'F'$ is equal to DEF.

Now, by § 160,

$$AE' : AC = AF' : AB,$$

and by hypothesis,

$$DE \text{ or } AE' : AC = DF : AB.$$

Hence, on account of the common ratio $AE' : AC$,

$$AF' : AB = DF : AB;$$

that is, AF' and DF are in the same ratio to AB, and are consequently equal.

In the same way it may be proved that $E'F'$ and EF, being in the same ratio to BC, are equal; and as the triangle DFE has its sides equal to those of $AE'F'$, it is equal to $AE'F'$, and is, therefore, similar to ABC.

181. *Theorem.* Lines AF, AG, &c. (fig. 100), drawn at pleasure through the vertex of a triangle, divide proportionally the base BC and its parallel DE, so that

$$DI : BF = IK . FG = KL : GH, \text{ \&c.}$$

Proof. Since DI is parallel to BF, the triangles ADI, ABF are equiangular, and give the proportion,

$$DI : BF = AI : AF;$$

also since IK is parallel to FG,

$$AI : AF = IK : FG;$$

and, therefore, on account of the common ratio $AI : AF$

$$DI : BF = IK : FG$$

It may be shown in like way, that

$$IK : FG = KL : GH, \text{ \&c.}$$

182. *Corollary.* When BC is divided into equal parts, the parallel DE is likewise divided into equal parts.

183. *Theorem.* The perpendicular AD (fig. 101) upon the hypothenuse BC of the right triangle BAC from the vertex A of the right angle, divides the triangle into two triangles BAD, CAD, which are similar to each other and to the whole triangle BAC.

Proof. *a.* The right triangles BAC and BAD are similar, by § 174, for the acute angle B is common to them both.

b. In the same way it may be shown, that DAC is similar to BAC, and, therefore, to BAD.

184. *Corollary.* From the similar triangles BAD, BAC, we have

$$BD : BA = BA : BC,$$

that is, the leg BA is a mean proportional between the hypothenuse BC and the adjacent segment BD.

a. In the same way AC is a mean proportional between BC and DC.

185. *Corollary.* From the similar triangles BAD, CAD, we have

$$BD : DA = DA : DC,$$

or, the perpendicular DA is a mean proportional between the segments BD, DC of the hypothenuse.

186. *Theorem.* If from a point A (fig. 102), in the circumference of a circle, a perpendicular AD is

drawn to the diameter BC, it is a mean proportional between the segments BD, DC of the diameter.

Proof. For, if the chords AB, AC are drawn, the triangle BAC is, by § 109, right-angled at A.

187. *Corollary.* The chord BA is a mean proportional between the diameter BC and the adjacent segment BD.

Likewise, AC is a mean proportional between BC and DC.

188. *Problem.* To find a mean proportional between two given lines.

Solution. Draw the indefinite line AB (fig. 103). Take AC equal to one of the given lines, and BC equal to the other. Upon AB as a diameter describe the semicircle ADB. At C erect the perpendicular CD, and CD is, by § 186, the required mean proportional.

189. *Theorem.* The parts of two chords which cut each other in a circle are reciprocally proportional, that is (fig. 104), $AO : DO = CO : BO$.

Proof. Join AD and CB. In the triangles AOD and COB, the angles AOD and COB are equal, by § 23; also the angles ADO and CBO are equal, by § 108, because they are each measured by half the arc AC, and, therefore, the triangles AOD and COB are similar by § 173, and give the proportion

$$AO : DO = CO : BO.$$

190. *Theorem.* If, from a point O (fig. 105), taken without a circle, secants OA, OD be drawn, the entire secants AO and DO are reciprocally proportional to the parts BO and CO without the circle, that is,

$$AO : DO = CO : BO.$$

Proof. Join AC and BD. In the triangles AOC and BOD, the angle O is common, and the angles BAC and BDC are equal, by § 108; these triangles are, therefore, similar, by § 173, and give the proportion

$$AO : DO = CO : BO.$$

191. *Theorem.* If, from a point O (fig. 106), taken without a circle, a tangent OC and a secant OA be drawn, the tangent is a mean proportional between the entire secant and the part without the circle, that is,

$$AO : CO = CO : BO.$$

Proof. When, in (fig. 105), the secant OC is turned about the point O until it becomes a tangent, as in (fig. 106), the points C and D must coincide, CO must be equal to DO, and the proportion (fig. 105)

$$AO : DO = CO : BO,$$

becomes (fig. 106) $AO : CO = CO : BO$.

192. *Problem.* To divide a given line AB (fig. 107) at the point C in *extreme and mean ratio*, that is, so that we may have the proportion

$$AB : AC = AC : CB.$$

Solution. At B erect the perpendicular BD equal to half of AB. Join AD, take DE equal to BD, and AC equal to AE, and C is the required point of division.

Proof. Describe the semicircumference EBF with the radius DB to meet AD produced in F; and, by the preceding proposition,

$$AF : AB = AB : AE;$$

and, by the theory of proportions,

$$AB : AF - AB = AE : AB - AE.$$

But $AB = 2.\ BD = EF$,

and $AE = AC$;

hence $AF - AB = AF - EF = AE = AC$,

and $AB - AE = AB - AC = BC$;

and the preceding proportion becomes

$$AB : AC = AC : BC.$$

193. *Theorem.* If two polygons $ABCD$, &c., $A'\,B'\,C'\,D'$, &c. (fig. 108) are composed of the same number of triangles ABC, ACD, &c., $A'B'C'$, $A'C'D'$, &c. which are similar each to each and similarly disposed, the polygons are similar.

Proof. Since the triangles ABC, &c. are similar to $A'B'C'$, &c., their angles must be equal each to each. Hence the angle A of the first polygon, which is the sum of the angles BAC, CAD, &c. is equal to the angle A' of the second polygon, which is the sum of $B'A'C'$, $C'A'D'$, &c. Also $B = B'$,

$$C = BCA + ACD = B'C'A' + A'C'D' = C', \text{ \&c.};$$

the polygons are therefore equiangular with respect to each other.

Their homologous sides are, moreover, proportional, for the similar triangles give

$$AB : A'B' = BC : B'C',$$

and $BC : B'C' = AC : A'C'$

$$= CD : C'D', \text{ \&c.}$$

Hence, by § 170, the polygons are similar

194. *Problem.* To construct a polygon similar to a given polygon $ABCD$, &c. (fig. 108) upon a given line $A'B'$, homologous to the side AB.

Equilateral similar Polygons are equal.

Solution. Join AC, AD, &c. Draw $A'C'$, $A'D'$, &c., making the angles $B'A'C = BAC$, $C'A'D = CAD$, &c.

Draw $B'C'$, making the angle $A'B'C' = ABC$, and meeting $A'C'$ at C'. Draw $C'D'$, making the angle $A'C'D' = ACD$, and meeting $A'D'$ at D'; and so on.

The polygon $A'B'C'D'$ &c. thus constructed, is the required polygon.

Proof. For, by § 173, the successive triangles $A'B'C'$, $A'C'D'$, &c. are similar to ABC, ACD, &c. each to each, and therefore, by the preceding theorem, the polygons are similar.

195. *Theorem.* If the similar polygons $ABCD$ &c. $A'B'C'D'$ &c. (fig. 109) have a side AB of the one equal to the homologous side $A'B'$ of the other, the polygons are equal..

Proof. The polygons are, by § 170, equiangular; they are also equilateral, for, by § 170, the ratio of BC to $B'C'$ is the same as that of AB to $A'B'$, or the ratio of equality; that is, $BC = B'C'$, and, in the same way $CD = C'D'$, &c.

If, then, $A'B'$ is placed upon AB, $B'C'$ will take the direction of BC, because the angle $B' = C'$; and C' will fall upon C, because $B'C' = BC$; and in the same way it may be shown, that D' falls upon D, E' upon E, &c; so that the polygons coincide, and are equal

196. *Theorem.* Two similar polygons $ABCD$ &c., $A'B'C'D'$ &c. (fig. 108), are composed of the same number of triangles ABC, ACD, &c., $A'B'C'$, $A'C'D'$, &c., which are similar each to each and similarly disposed

Proof. Construct upon $A'B'$ homologous to AB, by § 194, a polygon similar to $ABCD$ &c., and it must also be similar to $A'B'C'D'$ &c., and must therefore, by the preceding proposition, coincide with it; so that $A'B'C'D'$ &c. must, from § 194, be composed of triangles similar and similarly disposed to those of $ABCD$ &c.

197. *Theorem.* The perimeters of similar polygons are as their homologous sides.

Proof. From the definition of § 170, the similar polygons $ABCD$ &c. (fig. 108), $A'B'C'D'$ &c. give the pro portion

$$AB : A'B' = BC : B'C' = CD : C'D', \text{ \&c.}$$

Now the sum of the antecedents of this continued proportion is $AB + BC + CD +$ &c., or the perimeter of $ABCD$ &c., which we may denote by the letter P; and the sum of the consequents is $A'B' + B'C' + C'D' +$ &c., or the perimeter of $A'B'C'D'$ &c., which we may denote by P'.

Hence, from the theory of proportions,

$$P : P' = AB : A'B' = BC : B'C', \text{ \&c.}$$

198. *Theorem.* If two homologous sides AB, $A'B'$, (figs. 109, 110, 111) of two similar triangles, parallelograms, or trapezoids, are assumed as their bases, the altitudes CE, $C'E'$ are to each other as the homologous sides.

Proof. Since the acute angles CAB, $C'A'B'$ are, by § 170, equal, the right triangles AEC, $A'E'C'$ are, by § 174, similar, and give the proportion

$$AC : A'C' = CE : C'E'$$

199. *Corollary.* The homologous altitudes of two similar triangles, &c. are to each other as their homologous bases.

200. *Corollary.* The perimeters of two similar triangles, parallelograms, or trapezoids are to each other as their homologous altitudes.

CHAPTER XII.

REGULAR POLYGONS.

201. *Definition.* A *regular polygon* is one which is at the same time equiangular and equilateral.

Hence the equilateral triangle is the regular polygon of three sides, and the square the one of four.

202. *Theorem.* Every equilateral polygon, as *ABCD* &c. (fig. 112), which is inscribed in a circle, is regular.

Proof. As the polygon *ABCD* &c. is supposed to be equilateral, we have only to prove that it is also equiangular.

Now the arcs *AB*, *BC*, *CD*, &c. are equal, for they are subtended by the equal chords *AB*, *BC*, *CD*, &c. ; and, therefore, twice these arcs, or the arcs *ABC*, *BCD*, *CDE*, &c. are equal.

Hence the angles *ABC*, *BCD*, *CDE*, &c. are equal, since they are inscribed in equal segments.

203. *Theorem.* An infinitely small arc *AB* (fig. 113) coincides with its chord *AB*.

Proof. Through C the middle of the chord AB draw the radius DO. Complete the rectangle $DEAC$, by § 147; and, as the side DE is perpendicular to OD, it is a tangent to the circle.

The arc AD is, then, less than the sum of the including lines $AE + DE = AC + DC$; and

$$2\,AD < 2\,AC + 2\,DC,$$

or

$$\text{the arc } AB < \text{the chord } AB + 2\,DC;$$

or

$$\text{the arc } AB - \text{the chord } AB < 2\,DC,$$

that is, the difference between the arc AB and its chord is less than $2\,DC$.

But, by § 186,

$$CF \cdot AC = AC : CD = 2\,AC : 2\,CD$$
$$= \text{the chord } AB : 2\,CD;$$

that is, $2\,CD$ has the same ratio to the chord AB, which the infinitely small line AC has to CF; so that $2\,CD$ is infinitely small in comparison with the chord AB. And, as the difference between the chord and the arc is smaller than $2\,CD$, it must likewise be infinitely small in comparison with either the chord or the arc, and may, by § 99, be neglected. The arc AB is, therefore, equal to the chord AB, and must, by § 18 and 16, coincide with it.

204. *Theorem.* The circle is a regular polygon of an infinite number of sides.

Proof. Suppose the circumference $ABCD$ &c. (fig. 114) divided into the infinitely small and equal arcs AB, BC, CD, &c. The polygon formed by the chords of these arcs is, by § 202, a regular polygon of an infinite number of sides; but since, by the preceding theorem,

the arcs coincide with the chords, this polygon is the circle itself.

205. *Scholium.* The two preceding demonstrations contain the following obvious and necessary limitation of the axiom of § 99.

The infinitely small quantities, which are neglected by the axiom of § 99, must be infinitely small in comparison with those which are retained.

In the present case, indeed, the difference between the infinitely small arc and its chord is infinitely small, and yet it could not be neglected if it were not infinitely small in comparison with the arc. For, as the sum of all these differences corresponding to all the arcs of the circle has the same ratio to the sum of all the arcs, that is, to the entire circumference, which each difference has to its arc; the sum of the differences, that is, the difference between the circumference of the circle and the perimeter of the polygon of an infinite number of sides, would not be infinitely small, and, therefore, capable of being neglected, unless each difference were infinitely small in comparison with its arc.

206. *Theorem.* Two regular polygons $ABCD$, &c. $A'B'C'D'$, &c. (fig. 115), of the same number of sides, are similar.

Proof. For, they are equiangular with respect to each other, since the sum of their angles is the same, by § 72, and each angle of each polygon is found by dividing this common sum by the number of sides.

Their homologous sides are, moreover, proportional; for since

$$AB = BC = CD, \text{ \&c.}$$
$$A'B' = B'C' = C'D', \text{ \&c.}$$

we have

$$AB : AB' = BC : B'C' = CD : C'D', \text{ \&c.}$$

207. *Corollary.* Hence, and by § 197, the perimeters of regular polygons are to each other as their homologous sides.

208. *Theorem.* Two circles are similar regular polygons.

Proof. The number of sides of each circle is any infinite number whatever, and, if we choose, the same infinite number for all circles.

209. *Theorem.* A regular polygon of any number of sides may be inscribed in a given circle.

Proof. Suppose the circumference *ABCD* &c. (fig 116) to be divided into any number of equal arcs *AB*, *BC*, *CD*, &c. Their chords *AB*, *BC*, *CD*, &c. are also equal, by § 112; and the polygon *ABCD*, &c. formed by these chords is, by § 202, a regular polygon of a number of sides equal to that of the arcs *AB*, *BC*, *CD*, &c.

210. *Problem.* To inscribe in a given circle a regular polygon, which has double the number of sides of a given inscribed regular polygon *ABCD* &c. (fig. 116).

Solution. Bisect the arcs *AB*, *BC*, *CD*, &c. at the points *M*, *N*, *O*, *P*, &c. Join *AM*, *MB*, *BN*, *NC*, &c and *AMBNCO*, &c. is the required polygon.

Proof. For the sides *AM*, *MB*, *BN*, *NC*, &c. being the sides of equal arcs, are equal, and, by § 202, the polygon is regular.

211. *Corollary.* By bisecting the arcs *AM*, *MB*, *BN*, &c., a regular inscribed polygon is obtained of

4 times the number of sides of the given polygon; and, by continuing the process, regular inscribed polygons are obtained of 8, 16, 32, &c. times the number of sides of the given polygon.

212. ***Problem.*** **To inscribe a square in a given circle.**

Solution. Draw the two diameters AB and CD (fig. 117) perpendicular to each other; join AD, DB, BC, CA; and $ADBC$ is the required square.

Proof. The arcs AD, BD, BC, and AC are equal, being quadrants; and therefore their chords AD, DB, BC, and CA are equal, and, by §§ 201 and 202, $ADBC$ is a square.

213. *Corollary.* Hence, by §§ 210 and 211, a polygon may be inscribed in a circle of 8, 16, 32, 64, &c. sides.

214. ***Problem.*** To inscribe in a given circle a regular hexagon.

Solution. Take the side BC (fig. 118) of the hexagon equal to the radius AC of the circle, and, by applying it six times round the circumference, the required hexagon $BCDEFG$ is obtained.

Proof. Join AC, and we are to prove that the arc BC is one sixth of the circumference, or that the angle BAC is $\frac{1}{6}$ of four right angles, or $\frac{1}{3}$ of two right angles.

Now, in the equilateral triangle ABC, each angle, as BAC, is, by § 70, equal to $\frac{1}{3}$ of two right angles.

215. *Corollary.* Hence regular polygons of 12, 24, 48, &c. sides may, by §§ 210 and 211, be inscribed in a given circle

216. *Corollary.* An equilateral triangle BDF is inscribed by joining the alternate vertices, B, D, F.

217. *Problem.* To inscribe in a given circle a regular decagon.

Solution. Divide the radius AB (fig. 119) in extreme and mean ratio at the point C. Take BD for the side of the decagon equal to the larger part AC, and, by applying it ten times round the circumference, the required decagon $BDEF$ &c. is obtained.

Proof. Join AD, and we are to prove that the arc BD is $\frac{1}{10}$ of the circumference, or that the angle BAD is $\frac{1}{10}$ of four right angles, or $\frac{1}{5}$ of two right angles.

Join DC. The triangles BCD and ABD have the angle B common; and the sides BC and BD, which include this angle in the one triangle, are proportional to the sides BD and AB, which include the same angle in the other triangle. For, by § 192,

$$BC : AC = AC : AB;$$

but, by construction, BD is equal to AC, and, being substituted for it in this proportion, gives

$$BC : BD = BD : AB.$$

The triangles BCD and ABD are therefore similar, by § 179.

Now the triangle ABD is isosceles, and therefore BCD must also be isosceles; and the side DC is equal to BD, which is equal to AC; so that the triangle ACD is also isosceles.

We have, therefore,

$$\text{the angle } A = \text{the angle } ADC;$$

and, by § 71,

$$\text{the angle } BCD = \text{the angle } A + \text{the angle } ADC$$

$=$ twice the angle A.

But, in the isosceles triangles BCD and ACD,

the angle $BCD =$ the angle CBD

$=$ the angle ADB

$=$ twice the angle A,

and the sum of the three angles ABD, ADB, and A of the triangle ABD, or by § 65, two right angles, is equal to five times the angle A. Hence, A is $\frac{1}{5}$ of two right angles.

218. *Corollary.* Hence, regular polygons of 20, 40, 80, &c. sides may, §§ 210 and 211, be inscribed in a given circle.

219. *Corollary.* A regular pentagon $BEGIL$ is inscribed by joining the alternate vertices B, E, G, I, L.

220. ***Problem.*** To inscribe in a given circle a regular polygon of 15 sides.

Solution. Find, by § 217, the arc AB (fig. 120) equal to $\frac{1}{10}$ of the circumference, and, by § 214, the arc AC equal to $\frac{1}{6}$ of the circumference, and the chord BC, being applied 15 times round the circumference, gives the required polygon.

Proof. For the arc BC is $\frac{1}{6} - \frac{1}{10} = \frac{1}{15}$ of the circumference.

221. *Corollary.* Hence, regular polygons of 30, 60, 120, &c. sides may, by §§ 210 and 211, be inscribed in a given circle.

222. ***Problem.*** To circumscribe a circle about a given regular polygon $ABCD$ &c. (fig. 121)

Solution Find, by § 149, the circumference of a circle which passes through the three vertices A, B, C; and this circle is circumscribed about the given polygon.

Proof. Suppose the circumference divided into the same number of equal arcs AB', BC', &c. as that of the sides of the given polygon. The chords AB', $B'C'$, &c. form, by § 202, a regular polygon, which, by § 206, is similar to $ABCD$ &c.

Hence,

$$\text{the angle } ABC = \text{the angle } AB'C';$$

and, consequently, by § 108, the arc ABC, which is twice the arc AB, is equal to the arc $AB'C'$, which is twice the arc AB'. We have then,

$$\text{the arc } AB = \text{the arc } AB',$$

and the chord AB is equal to the chord AB', and coincides with it. The polygons $AB'C'D'$ &c., $ABCD$ &c., must, therefore, by § 195, coincide; and the circle is circumscribed about the given polygon.

223. *Corollary.* There is a point O in every regular polygon equally distant from all its vertices, and which is called *the centre of the polygon.*

224. *Corollary.* If we join AO, BO, CO, &c., the angles AOB, BOC, COD, &c. are all equal, and each has the same ratio to four right angles, which the arc AB has to the circumference.

225. *Corollary.* The isosceles triangles AOB, BOC, COD, &c. are all equal.

226. *Corollary.* The angles OAB, OBA, OBC, OCB, &c. are all equal, and each is half of the angle ABC.

227. *Problem.* To inscribe in a given circle a regu

lar polygon, similar to a given regular polygon *ABCD* &c. (fig. 123)

Solution. From the centre *O* of the given polygon draw the lines *AO*, *BO*; at the centre *O′* of the given circle make the angle *A′O′B′* equal to *AOB*, and the chord *A′B′*, being applied round the circumference as many times as *ABCD* &c. has sides, gives the required polygon *A′B′C′D′* &c., as is evident from § 224.

228. *Theorem.* The sides of a regular polygon are all equally distant from its centre.

Proof. Let fall the perpendiculars *OM*, *ON*, *OP*, &c. (fig. 122), from the centre *O*, upon the sides *AB*, *BC*, &c. In the right triangles *OAM*, *OBM*, *OBN*, *OCN*, *OCP*, &c., the hypothenuses *OA*, *OB*, *OC*, &c. are all equal, by § 223, and the legs *AM*, *MB*, *BN*, *NC*, *CP*, &c. are equal, since each is, by § 116, half of *AB*, or of its equal *BC*, &c. The triangles *OAM*, *OBM*, *OBN*, &c. are, consequently, equal, by § 64; and the perpendiculars *OM*, *ON*, *OP*, &c. are equal.

229. *Problem.* To inscribe a circle in a given regular polygon *ABCD* &c. (fig. 124).

Solution. From the centre *O* of the polygon, with a radius equal to *OM*, the distance of *AB* from *O*, describe a circle, and it is the required circle.

Proof. The distances *OM*, *ON*, *OP*, &c. are all equal, by § 228, and therefore the circumference passes through the points *M*, *N*, *P*, &c.; and the sides *AB*, *BC*, *CD*, &c. are all, by § 120, tangents to the circle; and the circle is, by § 118, inscribed in the polygon.

230. *Problem.* To circumscribe about a given circle a polygon similar to a given inscribed polygon *ABCD* &c. (fig. 125).

Solution. Through the points A, B, C, D, &c. draw the tangents $A'B'$, $B'C'$, $C'D'$, &c. and $A'B'C'D'$ is the required polygon.

Proof. The triangles $AB'B$, $BC'C$, &c. are by § 151, isosceles; they are also equal, for the sides AB, BC, &c. are equal, and the angles BAB', ABB', CBC', BCC', &c. are equal because they are measured by the halves of the equal arcs AB, BC, &c. Hence the angles A', B', &c. are equal, and the sides $A'B'$, $B'C'$, &c. are equal, and $A'B'C'$ &c. is a regular polygon of the same number of sides with ABC &c.

231. *Corollary.* A regular polygon of 4, 8, 16, &c.; 3, 6, 12, &c.; 5, 10, 20, &c.; 15, 30, 60, &c. sides; or, one similar to any given regular polygon may, therefore, be circumscribed about a circle by means of §§ 212–221, and 228.

232. *Theorem.* The homologous sides of regular polygons of the same number of sides are to each other as the radii of their circumscribed circles, and also as the radii of their inscribed circles.

Proof. Let $ABCD$, &c., $A'B'C'D'$, &c. (fig. 126) be regular polygons of the same number of sides, and let O, O' be their centres; OA, $O'A'$ are the radii of their circumscribed circles, and the perpendiculars OP, $O'P'$ are the radii of their inscribed circles.

Join OB, $O'B'$. The triangles OAB, $O'A'B'$ are similar, by § 173, for the angle $OAB = OBA = O'B'A' = O'A'B'$ for each is half the angle $ABC = A'B'C'$. Hence, by § 198,

$$OP : O'P' = AB : A'B' = OA : O'A'.$$

233. *Corollary.* Hence, the perimeters of regular polygons of the same number of sides are, by § 207, to each other as the radii of their inscribed circles, and also as the radii of their circumscribed circles.

234. *Theorem.* The circumferences of circles are to each other as their radii.

Proof. For circles are similar regular polygons, by § 208, and the radii of their inscribed and circumscribed circles are their own radii.

235. *Corollary.* The circumferences of circles are to each other as twice their radii, or as their diameters.

236. *Corollary.* If we denote the circumference of a circle by C, its radius by R, and its diameter by D; also the circumference of another circle by C', its radius by R', and its diameter by D', we have

$$C : C' = R : R' = D : D'.$$

Hence

$$C : R = C' : R';$$

and

$$C : D = C' : D'.$$

Hence, the circumference of every circle has the same ratio to its radius; and also to its diameter.

237. *Corollary.* If we denote the ratio of the circumference, C, of a circle to its diameter, D, by π, we have

$$C : D = \pi,$$

also

$$C = \pi \times D = 2\,\pi \times R.$$

and

$$D = C : \pi,$$

$$R = C : 2\,\pi$$

238. *Corollary.* π is the circumference of a circle whose diameter is unity, and the semicircumference of a circle whose radius is unity.

CHAPTER XIII.

AREAS.

239. *Definitions.* Equivalent figures are those which have the same surface.

The *area* of a figure is the measure of its surface.

240. *Definition.* The *unit of surface* is the square whose side is a linear unit; so that the area of a figure denotes its ratio to the unit of surface.

241. *Theorem.* Two rectangles, as $ABCD$, $AEFG$ (fig. 127) are to each other as the products of their bases by their altitudes, that is,

$$ABCD : AEFG = AB \times AC : AE \times AF.$$

Proof. *a.* Suppose the ratio of the bases AB to AE to be, for example, as 4 to 7, and that of the altitudes AC : AF to be, for example, as 5 to 3.

AE may be divided into 7 equal parts Aa, ab, bc, &c., of which AB contains 4; and, if perpendiculars aa', bb', &c. to AE are drawn through a, b, c, &c., the rectangle $ABCD$ is divided into 4 equal rectangles $Aa a' C$, $abb'a'$, &c., and the rectangle $AEFG$ is divided into 7 equal rectangles $Aaa''F$, $abb''a''$, &c.

Again, AC may be divided into 5 equal parts Am, mn, &c., of which AF contains 3; and, if perpendiculars mm',

nn', &c. to AC are drawn through m, n, &c., each of the partial rectangles of $ABCD$ is divided into 5 equal rectangles; and each of the partial rectangles of $AEFG$ is divided into 3 equal rectangles; and all these small rectangles are, evidently, equal.

Hence $ABCD$ contains 4×5 of them, and $AEFG$ contains 3×7 of them; that is,

$$ABCD : AEFG = 4 \times 5 : 7 \times 3,$$

which is equal to the product of the ratio $4 : 7$ by $5 : 3$, or of $AB : AE$ by $AC : AF$, so that

$$ABCD : AEFG = AB \times AC : AE \times AF.$$

b. This demonstration is readily extended to the case where the sides are incommensurate by dividing the rectangles into infinitely small rectangles.

242. *Corollary*. The rectangle $ABCD$ is, consequently, by § 240, to the unit of surface, as $AB \times AC$ to unity, or as the product of its base multiplied by its altitude to unity.

Hence the area of a rectangle $ABCD$ is the product of its base by its altitude.

243. *Corollary*. The area of a square is the square of one of its sides.

244. *Corollary*. Rectangles of the same altitude are to each other as their bases, and rectangles of the same base are to each other as their altitudes.

245. *Theorem*. Any two parallelograms $ABCD$, $ABEF$ (fig. 128) of the same base and altitude are equivalent.

Proof. The triangles ACF and BDE are equal, by § 51; for the sides AC and BD are equal, by § 78, being

the opposite sides of *ABCD*; also *AF* and *BE* are equal, being the opposite side of *ABFE*; and the angles *CAF* and *DBE* are equal, by § 29, since they have their sides parallel.

If, now, the triangle *ACF* is subtracted from the whole figure *ABCE*, the remainder is *ABFE*; and if *BDE* is subtracted from the whole figure, the remainder is *ABCD*. Hence, as *ABCD* and *ABFE* are the remainders, after taking equal triangles from the same figure, they must be equivalent.

246. *Corollary.* A parallelogram is equivalent to a rectangle of the same base and altitude.

247. *Corollary.* The area of a parallelogram is the product of its base by its altitude.

248. *Corollary.* Parallelograms of the same base are to each other as their altitudes; and those of the same altitude are to each other as their bases.

249. *Problem.* Every triangle is half of a parallelogram of the same base and altitude.

Proof. For the triangle *ABC* (fig. 39) is, by § 77, half of the parallelogram *ABCD* of the same base and altitude, and it is, therefore, by § 245, half of any parallelogram of the same base and altitude.

250. *Corollary.* All triangles of the same base and altitude are equivalent.

251. *Corollary.* The area of a triangle is half the product of its base by its altitude.

252. *Corollary.* Triangles of the same base are to each other as their altitudes, and triangles of the same altitude are to each other as their bases.

253. *Theorem.* The area of a trapezoid is half the product of its altitude by the sum of its parallel sides.

Proof. Draw the diagonal AD (fig. 129); the trapezoid $ABCD$ is divided into two triangles ACD and ABD, the bases of which are AB and CD, and the altitude of each is, by § 82, EF.

The area of ABD is, by § 251,

$$= \tfrac{1}{2}\, EF \times AB,$$

and the area of ACD

$$= \tfrac{1}{2}\, EF \times CD;$$

the sum of which is

$$\text{the trapezoid } ABCD = \tfrac{1}{2}\, EF \times (AB + CD).$$

254. *Lemma.* The line, which joins the middle points of the two sides of a trapezoid which are not parallel, is parallel to the two parallel sides, and is equal to half their sum.

Proof. a. Through the middle points H and I (fig. 129) of the sides AC and BD, draw HI, and through I draw OT parallel to CA.

The triangles DIO and ITB have the side DI equal to IB, the angle DIO equal to the vertical angle BIT, and the angle IDO equal, by § 30, to IBT; and, therefore, the triangles DIO and ITB are equal, by § 53; and

$$OI = IT = \tfrac{1}{2}\, OT.$$

But, in the parallelogram $OCAT$, we have, by § 78,

$$CA = OT;$$

whence

$$OI = \tfrac{1}{2}\, CA = CH;$$

so that $CHIO$ is, by § 81, a parallelogram; and HI is parallel to CD, and also to AB

b. Again, in the equal triangles DIO, TIB, we have

$$DO = TB\,;$$

whence

$$HI = CO = CD + DO,$$

and also

$$HI = AT = AB - BT = AB - DO\,;$$

the sum of which is

$$2\,HI = AB + CD,$$

or

$$HI = \tfrac{1}{2}\,(AB + CD).$$

255. *Corollary*. The area of a trapezoid is the product of its altitude by the line joining the middle points of the sides which are not parallel.

256. *Theorem*. The square described upon the hypothenuse of a right triangle is equivalent to the sum of the squares described upon the other two sides.

Proof. Let squares be constructed upon the three sides of the right triangle ABC (fig. 130), right-angled at B. From B let fall upon AC the perpendicular BDE, and the square $ACSR$ is divided into the two rectangles $ADER$ and $DCES$.

Now the area of $ADER$ is, by § 242, $AD \times AR = AD \times AC$; and the area of the square $ABNM$ is, by § 243, AB^2.

But, by § 184,

$$AD : AB = AB : AC\,;$$

or multiplying extremes and means,

$$AB^2 = AD \times AC\,;$$

that is, the square $ABNM$ is equivalent to the rectangle $ADER$.

Ratio of the Squares of the Sides of a Right Triangle.

It may be shown in the same way, that the square $BCPO$ is equivalent to the rectangle $DCSE$; and, therefore, the square $ACSR$ is equivalent to the sum of the squares $ABNM$ and $BCPO$, or

$$AC^2 = AB^2 + BC^2.$$

257. *Corollary.* The square of one of the legs of a right triangle is equivalent to the difference between the square of the hypothenuse and the square of the other leg; or

$$AB^2 = AC^2 - BC^2.$$

258. *Corollary.* In the square (fig. 117),

$$AB^2 = AD^2 + DB^2 = 2\ AD^2 = 2 \times ADBC;$$

or the square described upon the diagonal of a square is twice as great as the square itself.

Hence

$$AB^2 : AD^2 = 2 : 1,$$

and, extracting the square root,

$$AB : AD = \sqrt{2} : 1.$$

259. *Corollary.* Since (fig. 130),

$$AB^2 = AD \times AC,$$

we have

$$AC^2 : AB^2 = AC \times AC : AD \times AC = AC : AD;$$

and, in the same way,

$$AC^2 : BC^2 = AC : DC;$$

or the square of the hypothenuse of a right triangle is to the square of one of its legs, as the hypothenuse is to the segment of the hypothenuse adjacent to this leg, made by the perpendicular from the vertex of the right angle

260. *Corollary* Since

$$AB^2 = AD \times AC,$$

and

$$BC^2 = DC \times AC$$

we have

$$AB^2 : BC^2 = AD \times AC : DC \times AC = AD : DC;$$

or the squares described upon the two legs of a right triangle are to each other, as the adjacent segments of the hypothenuse made by the perpendicular from the vertex of the right angle.

261. *Problem.* To make a square equivalent to the sum of two given squares.

Solution. Construct a right angle C (fig. 131); take CA equal to a side of one of the given squares; take CB equal to a side of the other; join AB, and AB is a side of the square sought.

Proof. For, by § 256,

$$AB^2 = AC^2 + BC^2.$$

262. *Problem.* To make a square equal to the difference of two given squares.

Solution. Construct, by § 145, a right triangle, of which the hypothenuse BC (fig. 79) is equal to the side of the greater square, and the leg AB is equal to the side of the less square; and AC is the side of the required square.

Proof. For, by § 257,

$$AC^2 = BC^2 - AB^2.$$

263. *Problem.* To make a square equivalent to the sum of any number of given squares.

Solution. Take AB (fig. 132) equal to the side of one of the given squares. Draw BC, perpendicular to AB, and equal to the side of the second given square.

Join AC, and draw CD, perpendicular to AC, and equal to the side of the third given square

Join AD, and draw DE, perpendicular to AD, and equal to the side of the fourth given square; and so on. The line which joins A to the extremity of the last side is the side of the required square.

Proof. For, by § 256,

$$AC^2 = AB^2 + BC^2,$$
$$AD^2 = AC^2 + CD^2 = AB^2 + BC^2 + CD^2,$$
$$AE^2 = AD^2 + DE^2 = AB^2 + BC^2 + CD^2 + DE^2;\ \&c.$$

264. *Scholium.* If either of the squares BC^2, CD^2, &c. were to have been subtracted instead of being added, the problem might still have been solved by means of § 262.

265. *Problem.* To make a square which is to a given square in a given ratio.

Solution. Divide any line, as EG (fig. 133), by § 163, into two parts, at the point F, which are to each other in the given ratio of the given square to the required square.

Upon EG describe the semicircle EMG; draw FM perpendicular to EG.

Join ME and MG; take, on ME produced if necessary, $MH = AB$ the side of the given square.

Draw HI parallel to EG, meeting MG in I, and MI is the side of the required square.

Proof. Produce MF to P; and, as the triangle HMI is, by § 109, right-angled at M, we have, by § 260,

$$MH^2 : MI^2 = HP : PI.$$

But by § 181,

$$HP : PI = EF : FG;$$

whence, on account of the common ratio $HP : PI$,

$$MH^2 : MI^2 = EF : FG.$$

266. *Theorem.* Similar triangles are to each other as the squares of their homologous sides.

Proof. In the similar triangles ABC, $A'B'C'$ (fig. 109), we have, by § 199,

$$CE : C'E' = AB : A'B',$$

which, multiplied by the proportion

$$\tfrac{1}{2} AB : \tfrac{1}{2} A'B' = AB : A'B',$$

gives

$$\tfrac{1}{2} AB \times CE : \tfrac{1}{2} A'B' \times C'E' = AB^2 : A'B^2,$$

and, by § 251,

$$\text{the area of } ABC : \text{the area of } A'B'C' = AB^2 : A'B'^2.$$

267. *Corollary.* Hence, by § 197 & 198, similar triangles are to each other as the squares of their homologous altitudes, and as the squares of their perimeters.

268. *Theorem.* Similar polygons are to each other as the squares of their homologous sides.

Proof. In the similar polygons $ABCD$ &c., $A'B'C'D'$ &c. (fig. 108), the triangles ABC, $A'B\,C'$, which are similar, by § 196, give, by § 266, the proportion

$$ABC : A'B'C' = AC^2 : A'C'^2;$$

also the similar triangles ACD, $A'C'D'$, give the proportion

$$ACD : A\,C\,D' = AC^2 : A'C'^2;$$

hence, on account of the common ratio $AC^2 : A'C'^2$,

$$ABC : A'B'C = ACD : A'C'D'.$$

In the same way may be obtained the continued proportion

$$ABC : A'B'C' = ACD : A'C'D = ADE : A'D'E', \text{ \&c.}$$

Now the sum of the antecedents ABC, ACD, ADE, &c.

is the polygon $ABCD$ &c., and the sum of the consequents $A'B'C'$, $A'C'D'$, $A'D'E'$, &c. is the polygon $A'B'C'D'$ &c.; so that, by § 266,

$$ABCD \text{ \&c.} : A'B'C'D' \text{ \&c.} = ABC : A'B'C' = AB^2 : A'B'^2.$$

269. *Corollary.* Similar polygons are, therefore, to each other, by § 197, as the squares of their perimeters.

270. *Corollary.* As regular polygons of the same number of sides are, by § 206, similar polygons, they are to each other as the squares of their homologous sides, and, by § 232, as the squares of the radii of their inscribed circles, and also as the squares of the radii of their circumscribed circles.

271. *Theorem.* Circles are to each other as the squares of their radii.

Proof. For, by § 208, they are regular polygons of the same number of sides, and, as in § 234, the radii of their inscribed and circumscribed circles are their own radii.

272. *Problem.* Two similar polygons being given, to construct a similar polygon equivalent to their sum or to their difference.

Solution. Let A and B be the homologous sides of the given polygons. Find, by § 261, or by § 262, the line X such that the square constructed upon X is equal to the sum or the difference of the squares constructed upon A and B. The polygon similar to the given polygons, constructed, by § 194, upon the side X homologous to A or B, is the required polygon.

Proof. For, by § 268, the similar polygons construct-

ed upon A, B, and X, have the same ratio to each other as the squares upon A, B, and X.

273. *Corollary.* If A and B were the radii of two given circles, X would, by § 271, be the radius of a circle equivalent to their sum or to their difference.

274. *Corollary.* By the process of § 263, a polygon might be constructed equivalent to the sum of any number of given similar polygons, and similar to them, or a circle equivalent to the sum of any number of given circles; or, if either of the given polygons or circles is to be added instead of being substracted, the resulting polygon or circle may be obtained, as in § 264.

275. *Problem.* To construct a polygon similar to a given polygon, and having a given ratio to it.

Solution. Let A be a side of the given polygon. Find, by § 265, the side X of a square which is to the square, constructed upon A, in the given ratio of the polygons. The polygon, constructed upon X, similar to the given polygon, is the required polygon.

Proof. For, by § 268, the similar polygons constructed upon A and X, have the same ratio to each other as the squares upon A and X.

276. *Corollary.* In the same way, a circle may be constructed having a given ratio to a given circle, by taking for A and X the radii of the given and of the required circles.

277. *Theorem.* The area of any circumscribed polygon is half the product of its perimeter by the radius of the inscribed circle.

Proof. From the centre O (fig. 134) of the circle draw

OA, OB, OC, &c. to the vertices of the circumscribed polygon $ABCD$, &c. Draw the radii OM, ON, OP, &c. to the points of contact of the sides

If, now, the sides AB BC, CD, &c. are taken for the bases of the triangles OAB, OBC, OCD, &c.; their altitudes, being the radii OM, ON, OP, &c., are all equal. The area of each of these triangles is, then, by § 251, half the product of its base AB, BC, CD, &c. by the common altitude OM.

The sum of the areas of the triangles, or the area of the polygon is, consequently, half the product of the sum of the sides, AB, BC, &c. by the common altitude OM; that is, half the product of the perimeter $ABCD$ &c. of the polygon by the radius OM.

278. *Corollary.* Since a circle can, by § 229, be inscribed in any regular polygon, the area of the regular polygon is half the product of its perimeter by the radius of its inscribed circle.

279. *Theorem.* The area of a circle is half the product of its circumference by its radius.

Proof. For a circle is, by § 204, a regular polygon, and the radius of its inscribed circle is its own radius.

280. *Corollary.* If we use C, D, R, and π, as in § 237, and denote by A the area of a circle, we have

$$A = \tfrac{1}{2}.\ C \times R = \tfrac{1}{2}.\ 2\pi \times R \times R$$
$$= \pi \times R^2 = \tfrac{1}{4}\pi \times D^2$$

281. *Corollary.* When $R = 1$,

we have $A = \pi$

282. *Definition.* A *sector* is a part of a circle com-

prehended between an arc and the two radii drawn to its extremities, as AOB (fig. 135).

283. *Theorem.* The area of a sector is half the product of its arc by its radius.

Proof. Suppose the arc AB (fig. 135) of the sector AOB divided into the infinitely small arcs AM, MN, NP, &c. Draw the radii OM, ON, OP, &c.

The sector AOB is divided into the infinitely small sectors AOM, MON, NOP, &c.; which may, by § 203, be considered as triangles, having for their bases AM, MN, NP, &c., and for their altitudes the radii OA, OM, ON, &c.

The sum of the areas of these triangles, or the area of the sector is, then, half the product of the sum of the bases AM, MN, NP, &c. by the common altitude OA; that is, half the product of the arc AB by the radius AO.

284. *Corollary.* The area of the segment ADB is found by subtracting the area of the triangle AOB from that of the sector AOB.

285. *Scholium.* In order that no doubt may exist with regard to the accuracy of the demonstrations of § 283, 279, and 271, it is important to show that the infinitely small quantities, which are neglected in considering an infinitely small sector as a triangle with a base equal to its arc and an altitude equal to its radius, come within the limitation of § 205.

Now, the difference between the infinitely small sector AOB (fig. 113), and the triangle AOB, is the segment ADB. But the segment ADB is less than the rectangle $AEE B$; and, by § 242 and 251, the rectangle

$$\begin{aligned} AEE'B : \text{the triangle } AOB &= AB \times CD : \tfrac{1}{2} AB \times OC \\ &= CD : \tfrac{1}{2} OC \\ &= 2\, CD : OC; \end{aligned}$$

and, therefore, as 2 CD is infinitely small in comparison with OC, the rectangle $AEE'B$ and the segment ADB must be infinitely small in comparison with the triangle AOB, and may be neglected by § 204; so that the sector AOB is equivalent to the triangle AOB

The base of the triangle AOB is the chord AB, or, by § 203, the arc AB; and its altitude OC differs from the radius OD by the infinitely small quantity CD, which may be neglected.

The error arising from the neglect of these infinitely small quantities is altogether insensible, and cannot be rendered sensible by any magnifying process to which the mind can submit it; it is, then, no error at all. Indeed, if there be an error, suppose it to be represented by A. Since the aggregate of the quanties neglected is infinitely small, that is, *as small as we choose*; we can choose it to be less than the error A; a manifest absurdity, for the error cannot be greater than the aggregate of the quanties neglected, and yet we cannot escape this absurdity so long as we suppose the error A to be of any magnitude whatever.

286. *Definition. Similar sectors and similar segments* are such as correspond to similar arcs.

287. *Theorem.* Similar sectors are to each other as the squares of their radii.

Proof. The similar sectors AOB, $A'O'B'$ (fig. 136) are, by the same reasoning as in § 97, the same parts of their respective circles, which the angle $O = O'$ is of four right angles; and, therefore, they are to each other as these circles, or, by § 271, as the squares of the radii AO, $A'O'$.

288. *Theorem.* Similar segments are to each other as the squares of their radii.

Proof. Let ADB, $A'D'B'$ (fig. 136) be the similar segments. The triangles AOB and $A'O'B'$ are similar, by § 179; for $O = O'$; and, since $AO = BO$ and $A'O' = B'O'$, we have

$$AO : A'O' = BO : B'O'.$$

Hence, by § 266,

the triangle AOB : the triangle $A'O'B' = AO^2 : A'O'^2$;

also, by the preceding article,

the sector AOB : the sector $A'O'B' = AO^2 : A'O'^2$.

Hence, by the theory of proportions,

the sector AOB — the triangle AOB : the sector $A'O'B'$ — the triangle $A'O'B' = AO^2 : A'O'^2$;

that is,

the segment ADB : the segment $A'D'B' = AO^2 : A'O'^2$.

289. ***Problem.*** To find a triangle equivalent to a given polygon.

Solution. Let $ABCD$ &c. (fig. 137) be the given polygon. Join BD; through C, draw CM parallel to BD. Join DM, and $AMDE$ &c. is a polygon equivalent to the given polygon, and having the number of its sides less by one.

In the same way, a polygon may be found equivalent to $AMDE$, and having the number of its sides less by one; and by continuing the process, the number of sides may be at last reduced to three, and a triangle is obtained equivalent to the given polygon.

Proof. *a.* The number of sides of $AMDE$ &c. is less by one than that of $ABCD$ &c.; for the two sides AM, MD are substituted for the three sides AB, BC, CD, the other sides remaining unchanged.

b The polygon $AMDE$ &c. is equivalent to $ABCD$

&c.: for the part $ABDE$ &c. is common to both, and the triangles DBC, DBM are equivalent because they have the same base BD and the same altitude, by § 82, their vertices C and M being in the line CM parallel to this base.

290. *Problem.* To find a square equivalent to a given parallelogram.

Solution. Let B be the base and A the altitude of the given parallelogram. Find, by § 188, a mean proportional X between A and B, X is the side of the square sought.

Proof. For we have

$$A : X = X : B,$$

and, therefore,

$$X^2 = A \times B;$$

or, by §§ 242 and 243, the square constructed upon X is equivalent to the given parallelogram.

291. *Corollary.* A square may be found equivalent to a given triangle, by taking for its side a mean proportional between the base and half the altitude of the triangle.

292. *Corollary.* A square may be found equivalent to a given circumscribed polygon, by taking for its side a mean proportional between the perimeter of the polygon and half the radius of the inscribed circle.

293. *Corollary.* A square may be found equivalent to a given circle, by taking for its side a mean proportional between the radius and half the circumference of the circle.

294. *Corollary.* In general the *quadrature* of any given polygon may be found, that is, a square may be

found equivalent to any given polygon, by finding, by § 289, the triangle which is equivalent to the polygon, and, by § 291, the square which is equivalent to this triangle.

295. ***Problem.*** To construct a polygon equivalent to a given circle or polygon, P, and similar to a given polygon, Q.

Solution. Find, by §§ 293 or 294, M the side of a square equivalent to P, and N the side of a square equivalent to Q. Let A be one of the sides of Q. Find, by § 165, a fourth proportional X, to N, M, A. The polygon constructed by § 194, similar to Q upon X, homologous to A, is the required polygon.

Proof. Let Y be the polygon constructed upon X, we have only to prove that it is equivalent to P.

Now we have $\qquad N : M = A : X,$

whence $\qquad N^2 : M^2 = A^2 : X^2.$

Also, by § 268,

$$Q : Y = A^2 : X^2,$$

and leaving out the common ratio $A^2 : X^2$,

$$N^2 : M^2 = Q : Y.$$

But $\qquad N^2 = Q$ and $M^2 = P,$

whence $\qquad Q : P = Q : Y,$

or $\qquad Y = P.$

296. ***Problem.*** To construct a circle equivalent to a given polygon.

Solution. Find, by § 294, M the side of a square equivalent to the given polygon. Find, by § 265, R the side of a square which is to the given square in the ratio of the diameter of a circle to its circumference. R is the radius of the required circle.

Proof. Using π as in § 237, we have, by construction,

$$M^2 : R^2 = \pi$$

whence $\pi R^2 = M^2 =$ the given polygon.

That is, by § 280, the circle of which R is the radius is equal to the given polygon.

297. *Problem.* To construct a parallelogram, equivalent to a given square, and having the sum of its base and altitude equal to a given line.

Solution. Upon the given line AB (fig. 138) as a diameter describe a semicircle. At A, erect the perpendicular AC equal to the side of the given square. Draw CD parallel to AB, to meet the circumference at D. Draw DE perpendicular to AB; AE and EB are the required base and altitude.

Proof. For $AE + EB = AB$, and by § 290,

$$AE \times EB = DE^2 = AC^2.$$

298. *Problem.* To construct a parallelogram, equivalent to a given square, and having the difference of its base and altitude equal to a given line.

Solution. Upon the given line AB (fig. 139) as a diameter describe a circle. At A draw the tangent AC equal to the side of the given square. Through the centre O of the circle, draw the secant COE. CD and CE are the required base and altitude.

Proof. For we have

$$CE - CD = DE = AB,$$

and, by § 191,

$$CE : AC = AC : CD,$$

whence

$$AC^2 = CE \times CD$$

299. *Lemma.* If in a circle, whose radius is R, C is the chord of an arc and C' the chord of half the arc; C, C' and R will always satisfy the equation

$$C'^2 = 2\,R^2 - R\,\sqrt{(4\,R^2 - C^2)}.$$

Proof. Let AB (fig. 140) be the chord C and let AA' be C'; OMA' is, by § 117, perpendicular to AB, and the triangle OMA gives

$$OM^2 = OA^2 - AM^2 = R^2 - (\tfrac{1}{2}\,C)^2 = R^2 - \tfrac{1}{4}\,C^2$$

Hence, by § 187,

$$A'M = A'O - OM = R - \sqrt{(R^2 - \tfrac{1}{4}\,C^2)}$$

$$C'^2 = AA'^2 = A'M \times A'D'$$

$$= 2\,R^2 - 2\,R\,\sqrt{(R^2 - \tfrac{1}{4}\,C)}$$

$$= 2\,R^2 - R\,\sqrt{(4\,R^2 - C^2)}$$

300. *Corollary.* When $R = 1$, this equation becomes

$$C'^2 = 2 - \sqrt{(4 - C^2)}.$$

301. *Problem.* To find the ratio of the circumference of a circle to its diameter.

Solution. This ratio has been denoted, in § 237, by π; it does not admit of being expressed in numbers, and can only be obtained approximately. The principle of approximation consists in supposing the circumference to be equal to the perimeter of some one of its inscribed polygons: and the error of this hypothesis is the less, the greater the number of sides of the polygon

First Approximation. Let the radius AO (fig. 140) of the circle be unity, and its circumference is, by § 238, $2\,\pi$. If, now, the hexagon $ABCD$ &c. is inscribed in the circle, we have, by § 214, for its side,

$$AB = 1\,,$$

and for its perimeter

$$6 \times AB = 6;$$

so that, by supposing this perimeter to be equal to the circumference, we have for a first approximation

$$2\pi = 6, \text{ or } \pi = 3.$$

Second Approximation. Bisect the arcs AB, BC &c by the radii OA', OB' &c. Join AA', $A'B$ &c., and we have an inscribed polygon of 12 sides, and, by § 300,

$$AA'^2 = 2 - \sqrt{4 - AB^2};$$

$$AA' = \sqrt{(2 - \sqrt{4 - AB^2})}.$$

But $\quad AB = 1,$

whence $\quad AA' = \sqrt{2 - \sqrt{3}} = 0{\cdot}517$ nearly.

Hence the perimeter

$$AA'BB'C \text{ \&c.} = 12 \times AA' = 6{\cdot}204 \text{ nearly.}$$

And, if this is assumed for the circumference, we have, for the second approximation,

$$\pi = 3{\cdot}102 \text{ nearly.}$$

Third Approximation. If now we consider AB as the side of the inscribed polygon of 12 sides, AA' is the side of the polygon of 24 sides, and we have for AB,

$$AB = \sqrt{2 - \sqrt{3}} = 0{\cdot}517,$$

$$AB^2 = 2 - \sqrt{3} = 0{\cdot}267,$$

$$AA'^2 = 2 - \sqrt{4 - AB^2} = 2 - \sqrt{2 + \sqrt{3}} = 0{\cdot}068.$$

$$AA' = 0{\cdot}261.$$

The perimeter $\quad AA'B \text{ \&c.} = 24 \times AA' = 6{\cdot}26;$

and, by assuming this perimeter for the circumference, we have $\quad \pi = 3{\cdot}13$ nearly.

Further approximations might be obtained by supposing AB successively to be the side of an inscribed polygon of 24, 48, &c. sides, and by carrying the calculation to a

greater number of decimals. But it is useless to extend this process any further, as much more expeditious methods of calculating the value of π are obtained from the higher branches of mathematics, by means of which it has been calculated to 140 places of decimals.

For almost all practical purposes, the value of

$$\pi = 3 \cdot 1416,$$

is sufficiently accurate.

CHAPTER XIV.

ISOPERIMETRICAL FIGURES.

302. *Definitions.* Those figures which have equal perimeters are called *isoperimetrical figures.*

Among quantities of the same kind, that which is greatest is called a *maximum* ; and that which is smallest a *minimum.*

Thus the diameter of a circle is a *maximum* among all inscribed straight lines ; and a perpendicular is a *minimum* among all the straight lines drawn from a given point to a given straight line

303. *Theorem.* The maximum of isoperimetrical triangles of the same base is that triangle in which the two undetermined sides are equal.

Proof. Let the two triangles ACB and ADB (fig. 141) have the same base AB, and the same perimeter, that is,

$$AB + AC + BC = AB + AD + BD,$$

or, taking away AB,

$$AC + BC = AD + BD,$$

and suppose ACB isosceles, or $AC = CB$.

We are to prove that

$$\text{the triangle } ACB > \text{the triangle } ADB.$$

But, since these triangles have the same base AB, they are to each other as their altitudes CE and DF; so that we need only prove

$$CE > DF.$$

Produce AC to H, making $CH = CB = AC$. Join BH; and if a semicircle is described upon AH as a diameter with the radius $AC = CH$, it will pass through the point B; and ABH, being inscribed in it, must be a right angle.

Produce BH towards L, and take $DL = DB$. Join AL, and we have

$$AD + DL = AD + DB = AC + CB = AC + CH = AH.$$

But $$AD + DL > AL,$$

or $$AH > AL.$$

Hence, by § 41,

$$BH > BL,$$

and $$\tfrac{1}{2} BH > \tfrac{1}{2} BL.$$

Now, letting fall the perpendiculars CI and DM upon BH and BL, we have

$$\tfrac{1}{2} BH = BI = CE,$$

$$\tfrac{1}{2} BL = BM = DF;$$

whence $$CE > DF.$$

304. *Theorem.* The maximum of isoperimetrical polygons of the same number of sides is equilateral.

Proof. Let $ABCD$ &c. (fig. 142) be the maximum of isoperimetrical polygons of any given number of sides

Join AC. The triangle ABC must be the maximum of all the triangles which are formed upon AC, and with a perimeter equal to that of ABC. Otherwise a greater triangle AFC could be substituted for ABC, without changing the perimeter of the polygon, which would be inconsistent with the hypothesis that $ABCD$ &c. is the maximum polygon.

Therefore, by the preceding article,

$$AB = BC.$$

In the same way it may be proved, that

$$BC = CD = DE, \text{ \&c.}$$

305. *Theorem.* Of all triangles, formed with two given sides making any angle at pleasure with each other, the maximum is that in which the two given sides make a right angle.

Proof. Let ABC, ADC (fig. 143) be triangles, formed with the side AC common and the side $AB = AD$, and suppose BAC to be a right angle.

As these triangles have the same base AC, they are to each other as their altitudes AB and DE. But

$$AB = AD,$$

and, by § 39, $AD > DE$;

whence $AB > DE$,

and the triangle $ABC >$ the triangle ADC.

306. *Theorem.* The maximum of polygons formed of sides, all given but one, can be inscribed in a semicircle having the undetermined side for its diameter.

Proof. Let $ABCD$ &c. (fig. 144) be the maximum polygon formed of the given sides AB, BC, CD &c.

Draw from either vertex, as D, to the extremities A

and S of the side not given, the lines DA, DS. The triangle ADS must be the maximum of all triangles formed with the sides A and S; otherwise, either by increasing or else by diminishing the angle ADS, the triangle ADS would be enlarged, while the rest of the polygon $ABCD$, DEF &c. would be unchanged; so that the polygon would be enlarged, which is inconsistent with the hypothesis that it is the maximum polygon. The angle ADS is, therefore, a right angle by the preceding article, and is inscribed in the semicircle which has AS for its diameter.

307. *Theorem.* The maximum of all polygons formed of given sides can be inscribed in a circle.

Proof. Let $ABCD$ &c. (fig. 145) be a polygon which can be inscribed in a circle, and $A'B'C'D'$ &c. one which cannot be inscribed in a circle, but equilateral with respect to $ABCD$ &c.

Draw the diameter AM. Join EM, MF. Upon $E'F'$, equal to EF, construct the triangle $E'M'F'$, equal to EMF, and join $A'M'$.

The polygon $ABCDEM$, which is inscribed in the semicircle having AM for its diameter is, by the preceding article, greater than $A'B'C'D'E'M'$ formed of the same sides but one, and which cannot be so inscribed. In the same way

$$\text{the polygon } AMFG \text{ \&c.} > A'M'F'G' \text{ \&c.}$$

Hence, the entire polygon $ABCDEMF$ &c. $> A'B'C'D'$ $E'M'F'$ &c., and, subtracting the triangle $EMF = E'M'F'$

$$\text{the polygon } ABCD \text{ \&c.} > A'B'C'D' \text{ \&c.}$$

308. *Theorem.* The maximum of isoperimetrical polygons of the same number of sides is regular.

Proof. For, by § 304, it is equilateral; and, by the preceding article, it can be inscribed in a circle; so that, by § 202, it is regular.

309. *Theorem.* Of isoperimetrical regular polygons that is the greatest which has the greatest number of sides.

Proof. Let $ABCD$ &c., $A'B'C'D'$ &c. (fig. 146) be two isoperimetrical regular polygons, of which $ABCD$ &c. has the greater number of sides.

Denote the area of $ABCD$ &c. by S, and the radius OH of its inscribed circle by R; and denote the area of $A'B'C'D'$ &c. by S', and the radius $O'H'$ of its inscribed circle by R'; also the common perimeter of the two polygons by P.

Then we have, by § 277,

$$S : S' = \tfrac{1}{2} P \times R : \tfrac{1}{2} P \times R',$$

or, striking out the common factor $\frac{1}{2} P$,

$$S : S' = R : R';$$

so that, in order to prove

$$S > S',$$

we have only to prove

$$R > R'.$$

Upon $A'B'$, as a side, describe a polygon $A'B'C''D''$ &c. similar to $ABCD$ &c.; denote its perimeter by P', and the radius $O''M'$ of its inscribed circle by R''.

Join $A'O'$ and $A'O''$; describe the arc $M'N$ with the radius R', and the arc $M'N'$ with the radius R''.

The half side $A'M'$ is, evidently, the same part of the perimeter P, which the arc $M'N$ is of its circumference, which circumference is, by § 237, equal to $2\,\pi \times R'$; that is,

$$A'M' : P = M'N : 2\,\pi \times R',$$

and in the same way,

$$P'' : A'M' = 2\,\pi \times R'' : MN',$$

the product of these two proportions is, by striking out the factors common to the terms of each ratio,

$$P'' : P = R'' \times MN' : R' \times M'N'.$$

But, by § 233,

$$P'' : P = R'' : R,$$

and, on account of the common ratio $P'' : P$,

$$R'' : R = R'' \times M'N' : R' \times M'N',$$

which, multiplied by the identical proportion

$$R' : R'' = R' : R'',$$

gives, by striking out the common factors,

$$R' : R = M'N' : M'N',$$

so that we need only prove

$$M'N' > M'N,$$

in order to prove

$$R > R'.$$

Now, the angle $A'O'M'$ is obtained by dividing 360° by twice the number of sides of the polygon $A'B'C'D'$, &c., and the angle $A'O''M'$ is obtained by dividing 360° by twice the number of sides of the polygon $A'B'C''D''$ &c., but the second number of sides was supposed to be greater than the first, and, therefore,

$$\text{the angle } A'O''M' < \text{the angle } A'O'M' ;$$

and, therefore, as the angle $O''A'M'$ is, by § 69, the remainder after subtracting the angle $A'O''M'$ from 90°, it is greater than the angle $O'A'M'$ which remains after subtracting $A'O'M'$ from 90° ; and $O''A'M'$ includes $O'A'M'$; so that the radius $M'O''$ is greater than $M'O'$, and the circle described with $M'O''$ as a radius includes the circle described with $M'O'$ as a radius.

Join $N N'$; and upon the middle of $N N'$ erect a perpendicular meeting the tangent NT to the arc NM' at T, which it will do, for the angle TNN', being less than the right angle TNL, is acute.

Join $N'T$, and, by § 42,

$$NT = N'T.$$

But since the concave broken line TNM' is included by $TN'M'$, we have

$$TN' + N'M' > TN + NM',$$

whence, omitting TN' equal to TN,

$$N'M' > NM',$$

and, therefore, $$R > R',$$

and $$S > S'.$$

310. *Corollary.* As the circle is a polygon of an infinite number of sides, that is, of a greater number of sides than any other regular polygon, it is greater than any polygon of a finite number of sides which has a perimeter equal to the circumference of the circle

SOLID GEOMETRY.

CHAPTER XV.

PLANES AND SOLID ANGLES.

311. *Theorem.* Three points not in the same straight line determine the position of the plane in which they are situated.

Proof. For if any plane, passing through two of the points, is swung around the line joining these two points, until it comes to a position in which it passes through the third point, it must remain in this position. For swinging it any further must remove it from this third point.

312. *Corollary.* Only one plane can be drawn through three points not in the same straight line.

313. *Theorem.* The common intersection of two planes, which cut each other, is a straight line.

Proof. For, if any two of the points common to the two planes be joined by a straight line, this straight line must, by § 14, be in both of the planes; and no point out of this straight line can, by § 312, be in the two different planes at the same time.

314. When two planes cut each other, they form an *angle*, the magnitude of which does not depend

upon the extent, but merely upon the position of the planes.

315. *Theorem.* The angle of two planes, which cut each other, is measured by the angle of two lines drawn perpendicular to the common intersection of the two planes, at the same point, one in one of the planes, and one in the other.

Proof. In order to show the legitimacy of this measure we have only to prove that the angle of the two lines is proportional to the angle of the two planes.

Let AB (fig. 147) be the common intersection of the two planes; and let AC and AD be the two lines drawn in these planes perpendicular to the common intersection AB.

Let a third plane be drawn having also the common intersection AB with the two given planes, and let AE be drawn in this plane perpendicular to AB. We are to prove that the angle of the planes DAB and CAB is to that of the two planes EAB and CAB as DAC is to EAC.

For this purpose, suppose the angles of the planes to be to each other as any two whole numbers, and let the angle of the two planes CAB and aAB be their common divisor, Aa being perpendicular to AB. The angle CAa must be a common divisor of the two angles CAE and CAD; and it is shown by precisely the reasoning so often adopted, that the angles of the planes are to each other as CAD to CAE.

316. *Corollary.* When the angle CAD is a right angle, the planes are perpendicular to each other.

317. *Definitions.* A straight line is perpendicular to a plane, when it is perpendicular to every straight line drawn through its foot in the plane.

Reciprocally, the plane, in this case, is perpendicu lar to the line.

The *foot* of the perpendicular is the point in which it meets the plane.

318. *Theorem.* When a straight line is perpendicular to two straight lines drawn through its foot in a plane, it is perpendicular to every other straight line drawn through its foot in the plane, and, consequently, is perpendicular to the plane.

Proof. Let CAC', DAD' (fig. 148) be the two lines to which AB is perpendicular, and let EAE' be any other line drawn in the plane, we are to prove that BA is perpendicular to EAE'.

Take AC equal to AC', and AD equal to AD', join DC $D'C'$.

Turn $D'AC'E'$ around upon the point A, keeping AD' and AC' perpendicular to AB until AD' falls upon AD, and then AC' will fall upon AC, because the angle $D'A'C'$ is equal to DAC', $D'C'$ will fall upon DC, E' upon E, and AE' upon AE. Therefore, the angle BAE' is equal to the angle BAE, and each is, by § 20, a right angle.

319. *Corollary.* The perpendicular BA is less than any oblique line BE, and measures the *distance* of the point B, from the plane.

320. *Theorem.* Oblique lines drawn from a point to a plane at equal distances from the perpendicular are equal; and of two oblique lines unequally distant the more remote is the greater.

Proof. a. The oblique lines BC, BD, BE &c. (fig 149) at the equal distances AC, AD, AE &c. from the perpendicular BA are equal; for the triangles BAC,

BAD, *BAE*, &c. are equal, by § 51, since the angles *BAC*, *BAD*, *BAE*, &c. are equal, being right angles, the sides *AC*, *AD*, *AE* &c. are equal, and the side *BA* is common.

b. Since the oblique line *BC′* is drawn to the line *AC′* at a distance *AC′* greater than *AC* from the perpendicular *BA*, it is, by § 41, greater than *BC* or its equal *BD* or *BE*.

321. *Corollary.* All the equal oblique lines *BC*, *BD*, *BE* &c. terminate in the circumference *CDE*, drawn with *A* as a centre, and a radius equal to *AC*.

322. *Theorem.* If a line is perpendicular to a plane, every line which is parallel to this perpendicular, is likewise perpendicular to the plane.

Proof. Let *AB* (fig. 150) be the perpendicular to the plane, and let *CD* be parallel to *AB*, *CD* is likewise perpendicular to the plane, that is, to every straight line, as *DE*, drawn through its foot in that plane. For, if *BH* be drawn through the foot of *AB*, parallel to *DE*, the angle *ABH* is, by § 317, a right angle; but, by § 29, the angle *CDE* is equal to *ABH*, and is, also, a right angle.

323. *Corollary.* Hence straight lines, which are perpendicular to the same plane are parallel.

324. *Theorem.* If two planes are perpendicular to each other, the line, which is drawn in one of the planes perpendicular to their common intersection, must be perpendicular to the other plane.

Proof. Let the plane *MN* (fig. 151) be perpendicular to the plane *PQ*; and let *AB* be perpendicular to the common intersection *AP*, we are to prove that *AB* is perpendicular to *MN*.

Draw, in the plane *MN*, *AC* perpendicular to *AP*, *BAC*

must, by § 316, be a right angle. As *AB* is, therefore, perpendicular to both *AC* and *AP*, it is, by § 318, perpendicular to the plane *MN*.

325. *Corollary.* If two planes are perpendicular to each other, the straight line, drawn through any point of the common intersection perpendicular to one of the planes, must be in the other plane.

326. *Theorem.* If two planes are perpendicular to a third plane, their common intersection is also perpendicular to this third plane.

Proof. For, by the preceding article, the straight line *AB* (fig. 152) drawn through the common point *A* of the three planes, perpendicular to the third plane *MN*, must be in both of the planes *AP* and *AQ*, and must, therefore, be their common intersection.

327. *Theorem.* Two parallel lines are always in the same plane.

Proof. Draw a plane *MN* (fig. 153) perpendicular to one of the parallels *AB*, it must also, by § 322, be perpendicular to the other parallel *CD*; and if a plane is drawn through the two points *A* and *C*, perpendicular to *MN*, *AB* and *CD* must both, by § 325, be in this plane.

328. *Definitions. A straight line and a plane are parallel* when all the points of the straight line are equally distant from the plane.

Two planes are parallel, when all the points of one of the planes are equally distant from the other plane.

329. *Theorem.* A straight line and a plane are parallel, when they are perpendicular to the same straight line.

Proof. Let the straight line BC (fig. 154) and the plane MN be perpendicular to the same straight line AB; we are, by § 319, and 328, to prove that the perpendicular DC let fall from any point C of the line BC upon MN is equal to AB.

Join AD; AB and CD are parallel, by § 323, also AD is, by § 317, perpendicular to AB, and being in the plane of the parallels AB, CD, must, by § 35, be parallel to BC; so that $ABCD$ is a parallelogram, and its opposite sides AB and CD are equal, by § 78.

330. *Theorem.* If two planes are perpendicular to the same straight line, they are parallel.

Proof. Let the planes MN, PQ (fig. 155) be perpendicular to the line AB; we are, by § 328, to prove that the line CD, drawn from any point of PQ perpendicularly to MN, is equal to AB.

Join BC, and, as BC is, by § 317, perpendicular to AB, it is, by § 329, parallel to MN; and, therefore, CD is equal to AB.

331. *Theorem.* If a straight line is perpendicular to one of two parallel planes, it must also be perpendicular to the other.

Proof. Thus, if AB (fig. 155) is perpendicular to the plane MN, it must also be perpendicular to the plane PQ, which is parallel to MN.

For the plane drawn through B, perpendicular to AB, must be parallel to MN, and must therefore coincide with the plane PQ.

332. *Theorem.* If two planes are parallel to a third, they are parallel to each other.

Proof. For any line perpendicular to the third plane must, by the preceding article, be perpendicular to both

the other planes ; so that these other planes, being perpendicular to the same straight line, are parallel, by § 330.

333. *Theorem.* Two parallel lines, comprehended between two parallel planes, are equal.

Proof. Let the two parallel lines *AB, CD* (fig. 156) be included between the two parallel planes *MN, PQ*.

If the parallel lines are perpendicular to the parallel planes, they are equal, by § 328.

Otherwise, draw from the points *A* and *C* the lines *AE, CF*, perpendicular to *MN*; and join *BE, DF*.

The triangles *ABE, CDF* are equal, by § 53 ; for the sides *AE* and *CF* are equal, by § 328 ; the right angles *AEB* and *CFD* are equal ; and the angles *BAK* and *DCF* are equal, by § 29, because they have their sides parallel ; hence *AB* is equal to *CD*.

334. *Theorem.* The intersections of two parallel planes by a third plane are parallel lines.

Proof. Let the intersections of the plane *AD* (fig 156) with the parallel planes *MN, PQ* be *AC* and *BD*. Through *A* and *C*, in the plane *AD*, draw the parallel lines *AB, CD* ; these parallels are equal by the preceding article, and, therefore, by § 81, *ABCD* is a parallelogram, and *AC* is parallel to *BD*.

335. *Theorem.* If a straight line is parallel to another straight line drawn in a plane, it is parallel to the plane.

Proof. Let *AC* (fig. 156) be parallel to the line *BD* in the plane *MN*.

Through any point *A* of the line *AC*, let a plane *PQ* be drawn parallel to *MN*. The intersection of *PQ* with the plane *ABCD* is, by the preceding article, parallel to *BD* ; and, as it also passes through the point *A*, it must coincide with *AC*.

Now, since AC is in the plane PQ parallel to MN, all its points must, by § 328, be equally distant from MN, and it is therefore parallel to MN.

336. *Theorem.* Two straight lines, comprehended between three parallel planes, are divided into parts that are proportional to each other.

Proof. Let the line AC (fig. 157) meet the three parallel planes MN, PQ, RS at the points A, B, C; and let the line DF meet the same planes at D, E, F.

Join AF cutting the plane PQ at H; join AD, BH, HE, CF. The intersections BH and CF of the parallel planes PQ and RS with the plane ACF, are parallel, and give, by § 160, the proportion

$$AB : BC = AH : HF.$$

In like manner, the intersections HE and AD of the parallel planes PQ and MN with the plane FAD are parallel, and give the proportion

$$AH : HF = DE : EF.$$

Hence, on account of the common ratio $AH : HF$,

$$AB : BC = DE : EF.$$

that is, the lines AB and DF are divided proportionally at B and E.

337. *Definitions.* When three or a greater number of planes meet at a point, a *solid angle* is formed; as S (fig. 158) formed by the planes ASB, BSC, CSD, DSA.

The point of meeting, S, of the planes, is called the *vertex* of the angle.

338. *Theorem.* If a solid angle is formed by three plane angles, the sum of either two of these angles is greater than the third.

Proof. Let S (fig. 159) be a solid angle formed by the three plane angles ASB, BSC, and ASC, and let ASC, be the greatest of these plane angles. We need only prove that $ASC < ASB + BSC$.

Draw SD, making the angle CSD equal to CSB. Draw any line AC. Take SB equal to SD; join BC and BA. The triangles SCB and SCD are equal, by § 51, and $CD = CB$. But, by § 18,

$$AC < AB + BC.$$

and, subtracting $DC = BC$,

we have $AD < AB$.

Now in the two triangles ASD and ASB, the side SD is equal to SB, and AS is common; but the third side $AD < AB$, and therefore, by § 63,

$$ASD < ASB,$$

and, adding $CSD = CSB$

$$ASC < ASB + CSB.$$

339. *Theorem.* The sum of the plane angles, which form a solid angle, is always less than four right angles.

Proof. Draw a plane (fig. 160) cutting the solid angle S in $ABCDE$ &c. From any point O within $ABCD$ &c. draw AO, BO, CO, DO, &c.

The number of the triangles AOB, BOC, COD, &c. is the same as that of the triangles ASB, BSC, CSD, &c.; and therefore the sum of the angles of AOB, BOC, &c. is the same as that of ASB, BSC &c.

But, of the solid angle B, the sum of the angles ABS, SBC is, by the preceding article, grea er than the angle ABC, which is the sum of ABO, OBC, that is,

$$ABS + SBC > ABO + OBC;$$

and, in the same way,

$$BCS + SCD > BCO + OCD,$$

$$CDS + SDE > CDO + ODE, \text{ \&c.}$$

Hence $ABS + SBC + BCS + SCD +$ &c., or the sum of the angle at the bases of the triangles, which have their vertices at S, is greater than $ABO + OBC + BCO + OCD +$ &c., or the sum of the angles at the bases of the triangles which have their vertices at O.

If, then, these two sums of the angles at the bases of the triangles are subtracted from the common sum of all the angles of each set of triangles, the remaining sum of the angles which have their vertices at S must be less than the sum of the angles which have their vertices at O, or, by § 26, than four right angles.

340. *Theorem.* If two solid angles are respectively contained by three plane angles which are equal, each to each, the planes of any two of these angles in the one have the same inclination to each other as the planes of the homologous angles in the other.

Proof. Let the solid angles S, S' (fig. 161) be included by the plane angles $ASB = A'S'B'$, $ASC = A'S'C'$, $BSC = B'S'C'$.

Take $SA = S'A'$ of any length at pleasure. Draw AB, AC, perpendicular to SA, in the planes ASB and ASC; and draw $A'B'$, $A'C'$, perpendicular to $S'A'$ in the planes $A'S'B'$ and $A'S'C'$.

In the triangles ASB, $A'S'B$, the side $AS = A'S$, the angle $ASB = A'S'B'$; and the right angle $SAB = S'A'B'$; hence, by § 54, $AB = A'B'$ and $SB = S'B'$.

In the same way, it may be shown that $AC = A'C'$, $SC = S'C'$.

Join BC, $B'C'$, and, in the triangles SBC, $S'B'C'$, the angle $BSC = B'S'C'$, the side $SB = S'B'$, and the side $SC = S'C'$; hence, by § 52, $BC = B'C'$.

In the triangles ABC, $A'B'C'$ the three sides are respec-

tively equal, and, therefore, by § 61, the angle BAC, which, by § 315, measures that of the planes ASB, ASC is equal to $B'A'C'$, which measures the angle of the planes $A'S'B'$, $A'S'C'$.

In the same way, it may be shown that the angles of the other planes are equal ; some changes, easily made, are, however, required in the demonstration when either of the angles ASB, ASC is obtuse.

CHAPTER XVI.

SURFACE AND SOLIDITY OF SOLIDS.

341. *Definitions.* *Equivalent* solids are those which have the same bulk or magnitude.

A *lamina or slice* is a thin portion of a solid included between two parallel planes.

342. *Theorem.* If two solids have equal bases and heights, and if their sections, made by any plane parallel to the common plane of their bases, are equal, they are equivalent.

Proof. Let $ABCDEF$, $A'B'C'D'E'F'$ (fig. 162) be the two solids. Let MNO, $M'N'O'$ be two equal sections made by a plane parallel to the base, and let PQR, $P'Q'R'$ be two other equal sections made by a plane infinitely near the former plane, and parallel to it.

The infinitely thin laminæ $MNOPQR$, $M'N'O'P'Q'R'$ are equal ; for if $M'N'O'$ be applied to its equal MNO, $P'Q'R'$ must be infinitely near coincidence with its equal PQR ; and the laminæ themselves can differ from coincidence only by a quantity infinitely smaller than either of them, and which may, by § 99 and 205, be neglected

But by drawing a series of parallel planes, infinitely near each other, the given solids are divided into laminæ, which are respectively equal to each other ; and, therefore, their sums or the entire solids must be equivalent.

343. *Definitions.* Every solid bounded by planes is called a *polyedron.*

The bounding planes are called the *faces;* whereas the *sides* or *edges* are the lines of intersection of the faces.

344. *Definitions.* A polyedron of four faces is a *tetraedron*, one of six is a *hexaedron*, one of eight is an *octaedron*, one of twelve a *dodecaedron*, one of twenty an *icosaedron*, &c.

The tetraedron is the most simple of polyedrons, for it requires at least three planes to form a solid angle, and these three planes leave an opening, which is to be closed by a fourth plane.

345. *Definitions.* A *prism* is a solid comprehended under several parallelograms, terminated by two equal and parallel polygons, as ABC &c. FGH &c. (fig. 163).

The *bases of the prism* are the equal and parallel polygons, as ABC &c., and FGH &c.

The *convex surface of the prism* is the sum of its parallelograms, as $ABFG + BCGH +$ &c.

The *altitude of a prism* is the distance between its bases, as PQ.

346. *Definitions.* A *right prism* is one whose lateral faces or parallelograms are perpendicular to the bases, as ABC &c. FGH &c. (fig. 164).

In this case each of the sides AF, BG &c. is equal to the altitude.

347. *Definitions.* A prism is *triangular*, *quadrangular*, *pentagonal*, *hexagonal*, &c., according as its base is a triangle, a quadrilateral, a pentagon, a hexagon, &c.

348. *Definitions.* The prism, whose bases are regular polygons of an infinite number of sides, that is, circles, is called a *cylinder* (fig. 165).

The line OP, which joins the centres of its bases, is called the *axis of the cylinder.*

In the *right cylinder* (fig. 166) the axis is perpendicular to the bases, and equal to the altitude.

349. *Corollary.* The right cylinder (fig. 166) may be considered as generated by the revolution of the right parallelogram $OABP$ about the axis OP.

The sides OA and PB generate, in this case, the bases of the cylinder, and the side AB generates its convex surface.

350. *Definitions.* A prism whose base is a parallelogram (fig. 167) has all its faces parallelograms, and is called a *parallelopiped.*

When all the faces of a parallelopiped are rectangles, it is called a *right parallelopiped.*

351. *Definitions.* The *cube* is a right parallelopiped, comprehended under six equal squares.

The cube, each of whose faces is the unit of surface, is assumed as the *unit of solidity.*

352. *Definition.* The *volume*, *solidity*, or *solid contents* of a solid, is the measure of its bulk, or is its ratio to the unit of solidity.

353. *Theorem.* The area of the convex surface of

a right prism or cylinder is the perimeter or circumference of its base multiplied by its altitude.

Proof. a. The area of each of the parallelograms **ABFG**, **BCGH**, &c., which compose the convex surface of the right prism (fig. 164) is, by § 247, the product of its base **AB**, **BC** &c., by the common altitude **AF**; and the sum of their areas, or the convex surface of the prism, is the sum of these bases, or the perimeter **ABCD** &c., by the altitude **AF**.

b. This demonstration is extended to the right cylinder by increasing the number of sides to infinity.

354. *Theorem.* The section of a prism or cylinder made by a plane parallel to the bases is equal to either base.

Proof. a. Let **LMNO**, &c. (fig. 163) be a section of the prism made by a plane parallel to the bases. It follows, from § 334, that **LM** is parallel to **AB**, **MN** to **BC**, &c.; and, consequently, the angle **LMN** is equal to **ABC**, by § 29, the angle **NMO** to **BCD**, &c. Moreover, in the parallelograms **ABLM**, **BCMN**, &c., **AB** is equal to **LM**, **BC** to **MN**, &c., and the polygons **ABCD** &c., **LMNO**, &c. are equiangular and equilateral with respect to each other, and are, therefore equal, by § 195.

b. The demonstration is extended to the cylinder by increasing the number of sides to infinity.

355. *Corollary.* Hence, from § 342, two prisms or cylinders of equal bases and altitudes are equivalent.

356. *Corollary.* Any prism or cylinder is equivalent to a right prism or cylinder of the same base and altitude.

357. *Theorem.* Two right parallelopipeds are to each other as the products of their bases by their altitudes.

Proof. Let the two right parallelopipeds be *ABCD EFGH*, *AKLM NOPQ* (fig. 168), which we will denote by *AG* and *AP*.

Then, if the sides of the rectangles *ABCD* and *AKLM* are commensurable, the rectangles can, by § 241, be divided into equal rectangles ; and, if, through each of the vertices of these small rectangles, perpendiculars are erected to the plane *AL*, the parallelopipeds *AG* and *AP* are divided into smaller right parallelopipeds. All the parallelopipeds of *AG* are equivalent, by § 355, as well as all those of *AP* ; and the number of parallelopipeds in *AG* is equal to the number of rectangles in *ABCD* ; and the number of parallelopipeds in *AP* is equal to the number of rectangles in *AKLM*.

If now the altitudes *AE* and *AN* are commensurate, *AN* can be divided into equal parts, of which *AE* contains a certain number ; and, if, through the points of division of *AN*, planes are drawn parallel to the base *AL*, each of the partial parallelopipeds of *AG* and *AP* are divided into smaller equal parallelopipeds, and all these smallest parallelopipeds are equal to each other.

Now, the whole number of the smallest parallelopipeds contained in *AG* is the product of the number of rectangles in its base *ABCD* by the number of divisions of its altitude *AE*, and the number contained in *AP* is the product of the number of rectangles in its base *AKLM* by the number of divisions in its altitude *AN*. Hence

$$AG : AP = ABCD \times AE : AKLM \times AN.$$

This demonstration is readily extended to the case where the sides are incommensurate, by dividing the solids into infinitely small parallelopipeds

358. *Corollary.* The solidity of any right parallelopiped or its ratio to the unit of solidity is, by § 352, the product of its base by its altitude, that is,

$$AG = ABCD \times AE.$$

359. *Corollary.* Since, by § 242,

$$ABCD = AB \times AD,$$

we have

$$AG = AB \times AD \times AE;$$

or the solidity of a right parallelopided is the product of its three dimensions.

360. *Corollary.* The solidity of a cube is the cube of one of its sides.

361. *Corollary.* Since, by § 356, any parallelopiped of a rectangular base is equivalent to a right parallelopiped of the same base and altitude, the solidity of any parallelopiped of a rectangular base is the product of its base by its altitude.

362. *Theorem.* The solidity of any parallelopiped is the product of its base by its altitude.

Proof. Any parallelopiped which has $ABCD$ (fig. 169) for its base is, by § 356, equivalent to the parallelopiped AG, which has the same base, and its sides AH, BE, CG, DF, perpendicular to the base $ABCD$.

But any other face may as well be assumed for the base of AG as $ABCD$; taking, then, the rectangle $ABEH$ for the base, the parallelopiped AG is, by § 361, equal to the right parallelopiped of the same base and altitude, that is, by drawing DK perpendicular to AB,

$$AG = DK \times ABEH = DK \times AB \times AH.$$

But, $ABCD = DK \times AB$;
hence $AG = ABCD \times AH$.

363. *Corollary.* Any two parallelopipeds of equivalent bases and the same altitude are equivalent.

364. *Corollary.* Parallelopipeds of the same base are to each other as their altitudes, and parallelopipeds of the same altitude are to each other as their bases.

365. *Theorem.* The solidity of a triangular prism is the product of its base by its altitude.

Proof. Let $ABC\ DEF$ (fig. 170) be a triangular prism.

Draw BG parallel to AC, CG parallel to AB, GH parallel to AD, meeting the plane EDF in H. Join EH, FH; AH is, evidently, a parallelopiped; and $BCG\ EFH$ is a triangular prism.

The triangular prisms $ABC\ DEF$ and $BCG\ EFH$ are equivalent, by § 355; since their altitude is the same and their bases ABC and BCG are equal, by § 77. Hence each of the prisms is half of the parallelopiped AH, and has half its measure, or the product of $\frac{1}{2}\ ABCG$ by the altitude, that is, the product of its own base by its altitude.

366. *Theorem.* The solidity of any prism or cylinder whatever is the product of its base by its altitude.

Proof. a. The prism ABC &c. FGH &c. (fig. 163) may be divided into the triangular prisms $ABC\ FGH$, $ACD\ FHI$ &c. by the planes $ACFH$, $ADFI$ &c., and, by the preceding section, the solidity of each of these triangular prisms is the product of its base ABC, ACD, &c. by the altitude PQ. Hence, the sum of these prisms or the entire prism is the product of the sum of the bases

by PQ, or of the entire base $ABCD$ &c. by the altitude PQ.

b. This demonstration is extended to cylinders by increasing the number of sides to infinity.

367. *Corollary.* Prisms or cylinders of equivalent bases and equal altitudes are equivalent.

368. *Corollary.* Prisms or cylinders of equivalent bases are to each other as their altitudes; and those of the same altitude are to each other as their bases.

369. *Corollary.* Denoting by R the radius, and by A the area of the base of a cylinder; and using π as in § 237, we have, by § 280,

$$A = \pi \times R^2.$$

Denoting, also, by H the altitude, V the solidity of the cylinder, we have, by § 366,

$$V = A \times H = \pi \times R^2 \times H.$$

370. *Definitions.* A *pyramid* is a solid formed by several triangular planes proceeding from the same point, and terminating in the sides of a polygon, as $SABCD$ &c. (fig. 171).

The point S is the *vertex of the pyramid.*

The polygon $ABCD$ &c. is the *base of the pyramid.*

The *convex surface of the pyramid* is the sum of the triangles $SAB + SAC$, &c.

The *altitude of the pyramid* is the distance of its vertex from its base.

371. *Definitions.* A pyramid is *triangular*, *quadrangular*, &c., when the base is a triangle, a quadrilateral, &c.

372. *Definitions.* A pyramid is *regular*, when the base is a regular polygon, and the perpendicular let fall from the vertex upon the base, passes through the centre of the base (fig. 172).

This perpendicular from the vertex is called the *axis* of the pyramid.

373. *Definitions.* When the base of a pyramid is a regular polygon of an infinite number of sides, that is, a circle, it is called a *cone* (fig. 173).

The *axis of the cone* is the line drawn from the vertex to the centre of the base.

A *right cone* is one the axis of which is perpendicular to the base (fig. 174).

374. *Corollary.* The right cone (fig. 174) may be considered as generated by the revolution of the right triangle SOA about the axis SO.

The leg OA, in this case, generates the base, and the hypothenuse SA, which is called the *side of the cone*, generates the *convex surface*.

375. *Theorem.* The area of the convex surface of the regular pyramid is half the product of the perimeter of the base by the altitude of one of the triangles.

Proof. The triangles SAB, SBC, &c. (fig. 172) are all equal, for, by § 201,

$$AB = BC = CD, \text{ \&c. };$$

and, since the oblique lines AS, SB, SC, &c., are all at equal distances OA, OB, OC, &c., from the perpendicular SO, they are equal by § 320. Hence the altitudes SH, SI, SK, &c. of these triangles are equal; and the sum of the areas of the triangles is half the product of the sum of their bases AB, BC, CD, &c. by the common

altitude SH; that is, the convex surface of the pyramid is half the product of the perimeter of its base by the altitude of one of its triangles.

376. *Corollary.* When the base of the regular pyramid is a polygon of an infinite number of sides, the pyramid is a right cone, and the altitude of each triangle becomes the side SA (fig. 174) of the cone.

Hence the area of the convex surface of the right cone is half the product of the circumference of the base by the side.

377. *Theorem.* The section of a pyramid made by a plane parallel to the base is a polygon similar to the base.

Proof. Let $MNOP$ &c. (fig. 171) be the section of a pyramid made by a plane parallel to its base $ABCD$ &c

Since MN is, by § 334, parallel to AB, we have

$$SB : SN = AB : MN,$$

and since NO is parallel to BC, we have

$$SB : SN = BC : NO;$$

and, on account of the common ratio, $SB : SN$,

$$AB : MN = BC : NO.$$

In the same way we might prove

$$AB : MN = BC : NO = CD : OP, \text{ \&c.}$$

whence the sides of the polygons $ABCD$ &c., $MNOP$ &c. are proportional.

The angles of the polygons are also equal; indeed on account of the parallel sides, we have

$$MNO = ABC,\ NOP = BCD, \text{ \&c.}$$

The polygons are therefore similar, by § 170.

378. *Corollary.* The section of a cone made by a plane parallel to the base is a circle.

379. *Corollary.* If the perpendicular ST is let fall from S upon the base, meeting the section at R, we have, by §§ 268 and 336,

$$ABCD \text{ \&c.} : MNOP \text{ \&c.} = AB^2 : MN^2 = SA : SM^2$$
$$= ST^2 : SR^2,$$

or, the base of a pyramid or cone is; to the section made by a plane parallel to the base, as the square of the altitude of the pyramid is to the square of the distance of the section from the vertex.

380. *Corollary.* If two pyramids or cones have the same altitude and their bases in the same plane, their sections made by a plane parallel to the plane of their bases are to each other as their bases.

If the bases are equivalent, the sections are equivalent.

If the bases are equal, the sections are equal.

381. *Theorem.* Two pyramids or two cones which have equal bases and altitudes are equivalent.

Proof. For, if their bases are placed in the same plane, their sections made by a plane parallel to the plane of their bases are equal; and, therefore, by § 342, the pyramids are equivalent.

382. *Theorem.* A triangular pyramid is a third part of a triangular prism of the same base and altitude.

Proof. From the vertices B, C (fig. 175) of the triangular pyramid $S\ ABC$, draw BD, CE parallel to SA. Draw SD, SE parallel to AB, AC, and join CE; $ABC\ SDE$ is a triangular prism.

The quadrangular pyramid $S\ BCED$ is divided by the plane SBE into two triangular pyramids $S\ BED$ $S\ BEC$, which are equivalent; for their bases BED

BEC are equal, by § 77 ; and their common altitude is the distance of their common vertex *S* from the plane of their bases.

Again, if the plane *SED* is taken for the base of *S BED* and the point *B* for its vertex, the pyramid *B SDE* is equivalent to *S ABC* ; for their bases *SED*, *ABC* are equal, and their common altitude is the altitude of the prism.

But the sum of the three equal pyramids *S ABC*, *S BED*, *S BEC* is the prism *ABC SDE*, and, therefore, either pyramid, as *S ABC*, is a third part of the prism.

383. *Corollary.* The solidity of a triangular pyramid is a third of the product of its base by its altitude.

384. *Theorem.* The solidity of any pyramid is one third of the product of its base by its altitude.

Proof. The planes *SAC*, *SAD*, &c. (fig. 171) divide the pyramid *S ABCD* &c. into triangular pyramids, the common altitude of which is the altitude of the entire pyramid. Hence the solidity of the entire pyramid is one third of the product of the sum of their bases *ABC*, *ACD*, &c., by the common altitude, that is, one third of the entire base by the altitude of the pyramid.

385. *Corollary.* The solidity of a cone is one third of the product of its base by its altitude.

386. *Corollary.* Pyramids or cones are to each other as the products of their bases by their altitudes.

387. *Corollary.* Pyramids or cones of the same altitude are to each other as their bases ; and those of equivalent bases are to each other as their altitudes.

388. *Corollary.* Pyramids or cones of equivalent bases and equal altitudes are equivalent.

389. *Corollary.* Any pyramid or cone is a third part of a prism or cylinder of the same base and altitude.

390. *Corollary.* Denoting by R the radius of the base of a cone, by H its altitude, by V its solidity, and using π, as in § 237, we have, by §§ 369 and 389,

$$V = \tfrac{1}{3}\,\pi \times R^2 \times H.$$

391. *Definitions.* A *truncated prism* is the portion of a prism cut off by a plane inclined to its base, as $ABC\ DEF$ (fig. 176).

The *base of the truncated prism* is the same as the base of the prism from which it is cut.

392. *Theorem.* A truncated triangular prism is equivalent to the sum of three pyramids, which have for their common base the base of the prism, and for their vertices the three vertices of the inclined section.

Proof. Draw the plane FAC (fig. 176), cutting off from the truncated triangular prism $ABC\ DEF$ the pyramid $FABC$, which has ABC for its base, and F for its vertex.

There remains the quadrangular pyramid $FACDE$, which the plane FEC divides into the two triangular pyramids $FAEC$ and $FCDE$.

Now $FAEC$ is equivalent to the pyramid $BAEC$, which has the same base AEC, and the same altitude, because the vertices F, B are in the line FB parallel to this base. But ABC may be taken for the base of $EABC$, and E for its vertex.

Lastly,

$$\text{the pyramid } FECD = \text{the pyramid } BECD,$$

for they have the same base ECD, and the same altitude

because their vertices F, B are in the line FB parallel to this base. Also, taking E as the vertex of $BECD$

$$\text{the pyramid } EBCD = \text{the pyramid } ABC'D,$$

for, they have the common base BDC, and their vertices A, E are in the line AE parallel to this base. But ABC may be taken for the base of $ABCD$, and D for its vertex.

Hence the truncated prism is equivalent to the sum of three pyramids, which have the common base ABC, and for their vertices E, F, and D.

393. *Definitions.* If a pyramid or cone is cut by a plane parallel to its base, the portion which remains after taking away the smaller pyramid or cone, is called the *frustum of a pyramid or cone*, as $ABCD$ &c. $MNOP$ &c. (fig. 171.)

The *convex surface* of the frustum of a pyramid is the sum of the trapezoids which compose its lateral faces.

The polygons $ABCD$ &c., $MNOP$ &c. are the *bases of the frustum*, and the distance between its bases is its altitude.

394. *Corollary.* The frustum of the right cone (fig. 174) may be considered as generated by the revolution of the trapezoid $OO'A'A$ about the side OO'.

The side AA', which is called the *side of the frustum*, in this case, generates *the convex surface.*

395. *Theorem.* The area of the convex surface of the frustum of a regular pyramid is half the product of the sum of the perimeters of its bases by the altitude of either of its trapezoids.

Proof. The trapezoids $ABMN$, $BCNO$, &c. (fig. 172)

are all equal; and the area of each is half the product of the sum of its parallel sides by their common altitude HH'. The sum of their areas, or the area of the convex surface of the frustum is, therefore, half the product of this common altitude, by the sum of all the parallel sides, that is, by the sum of the perimeters of the bases of the frustum.

396. *Corollary.* If a section $M'N'O'P'$ &c. is made by a plane parallel to the bases, and passing through the middle point R' of the altitude, it must, by § 336, bisect the lines AM, BN, &c.; and the area of each trapezoid is, by § 255, the product of its altitude by the line $M'N'$, $N'O'$, &c.

The area of the convex surface of the frustum is, therefore, the product of the altitude by the sum of these lines, that is, by the perimeter of the section made by the plane which bisects the lateral sides of the frustum.

397. *Corollary.* The area of the convex surface of the frustum of a right cone is half the product of its side by the sum of the circumferences of the bases; or it is the product of the side by the circumference of the section parallel to the bases which bisects the side.

398. *Theorem.* The area of the surface, described by a line revolving about another line in the same plane with it as an axis, is the product of the revolving line by the circumference described by its middle point.

Proof. a. If the revolving line is parallel to the axis, as in (fig. 166), it describes the convex surface of a right cylinder, the area of which is, by § 353, the product of

the circumference of the base by the altitude. But the altitude is equal to the revolving line, and the circumference of the base is, by § 354, equal to the circumference described by the middle point; and, therefore, in this case, the area of the surface described is the product of the revolving line by the circumference described by its middle point.

b. If the revolving line is inclined to the axis without meeting it, the surface described is the convex surface of the frustum of a right cone; and its area is as, in § 397, the product of the revolving line by the circumference described by its middle point.

c. When the revolving line meets the axis without cutting, the surface described is the convex surface of a right cone, and is included in the preceding case by considering it as a frustum whose upper base is the vertex of the cone.

399. *Scholium.* The case, where the revolving line cuts the axis, is not included in the preceding theorem.

400. *Theorem.* The *frustum* of a pyramid or cone is equivalent to the sum of three pyramids or cones, which have for their common altitude the altitude of the frustum, and whose bases are the lower base of the frustum, its upper base, and a mean proportional between them.

Proof. Let $ABCD$ &c. $MNOP$ &c. (fig. 171) be the given frustum. Denote the area of the lower base $ABCD$ &c. by V, and that of the upper base $MNOP$ &c. by V'; and denote the altitude ST of the greater pyramid by H, the altitude SR of the less pyramid by H', and the altitude RT of the frustum by H''.

Since the frustum is the difference between the pyramids, we have for its solidity, by § 384,

Solids of a Frustum of a Pyramid or Cone.

$$\tfrac{1}{3} V \times H - \tfrac{1}{3} V' \times H',$$

and, for the sum of three pyramids, which have H'' for their altitude and for their bases V, V' and the mean proportional $\sqrt{VV'}$ between V and V',

$$\tfrac{1}{3} H'' \times (V + V' + \sqrt{VV'})$$

$$= \tfrac{1}{3} H'' \times V + \tfrac{1}{3} H'' \times V' + \tfrac{1}{3} H'' \times \sqrt{VV'},$$

and we are to prove that these solidities are equal, or that

$$V \times H - V' \times H' = V \times H'' + V' \times H'' + \sqrt{VV'} \times H''.$$

Now $$H'' = H - H',$$

and, by § 379,

$$V : V' = H^2 : H'^2,$$

whence $$\sqrt{V} : \sqrt{V'} = H : H',$$

and, multiplying extremes and means

$$\sqrt{V} \times H' = \sqrt{V'} \times H.$$

If we multiply this equation successively by $\sqrt{V}$ and $\sqrt{V'}$ we obtain, by transposing the members of the first product,

$$\sqrt{VV'} \times H = V \times H',$$

$$\sqrt{VV'} \times H' = V' \times H;$$

the difference between which is

$$\sqrt{VV'} \times (H - H') = V \times H' - V' \times H, \text{ or}$$

$$\sqrt{VV'} \times H'' = V \times H' - V' \times H.$$

And if we add to this last equation the equations

$$V \times H'' = V \times H - V \times H'$$

$$V' \times H'' = V' \times H - V' \times H',$$

we get, by cancelling the terms which destroy each other,

$$\sqrt{VV'} \times H'' + V \times H'' + V' \times H'' = V \times H - V' \times H',$$

which is the equation to be proved, and the solidity of the frustum is therefore equal to

$$\tfrac{1}{3} H'' \times (V + V' + \sqrt{VV'}).$$

Solidity of the Frustum of a Cone.

401. *Corollary.* If R is the radius of the lower base of the frustum of a cone, and R' the radius of the upper base, we have, by § 280,

$$V = \pi \times R^2$$
$$V' = \pi \times R'^2,$$

hence

$$\sqrt{VV'} = \sqrt{\pi^2 \times R^2 \times R'^2} = \pi \times R \times R',$$

and the solidity of the frustum is

$$\tfrac{1}{3}\pi \times H' \times (R^2 + R'^2 + R \times R').$$

402. *Scholium.* The solidity of any polyedron may be found by dividing it into pyramids.

CHAPTER XVII.

SIMILAR SOLIDS.

403. *Definition.* *Similar polyedrons* are those in which the homologous solid angles are equal, and the homologous faces are similar polygons.

404. *Corollary.* Hence, from § 170, the sides of similar polyedrons are proportional to each other.

405. *Corollary.* From § 268, the faces of similar polyedrons are to each other as the square of their homologous sides; and, from the theory of proportions, the sums of the faces, or the entire surfaces of the polyedrons are also to each other as the squares of the homologous sides.

406. *Corollary.* The bases of similar prisms or pyramids are to each other as the squares of their altitudes; and the perimeters of their bases are to each other as their altitudes.

407. *Corollary.* The bases of *similar cylinders or cones* are to each other as the squares of their altitudes; and their altitudes are to each other as the circumferences of the bases, or as the radii of the bases.

408. *Corollary.* The convex surfaces of similar prisms, pyramids, cylinders, or cones are to each other as their bases, or as the squares of their altitudes.

409. *Corollary.* The convex surfaces of similar prisms or pyramids are to each other as the squares of their homologous sides.

410. *Corollary.* The convex surfaces of similar cylinders or cones are to each other as the squares of the radii of their bases.

411. *Theorem.* Similar prisms, pyramids, cylinders, or cones are to each other as the cubes of their altitudes.

Proof. Prisms, pyramids, cylinders, or cones are to each other, by § 366 and 386, as the products of their bases by their altitudes. But where these solids are similar, their bases are to each other, by § 406 and 407, as the squares of their altitudes; and the products of the bases by their altitudes, or their solidities are to each other, as the products of the squares of their altitudes by their altitudes, or as the cubes of their altitudes.

412. *Corollary.* Similar prisms or pyramids are to each other as the cubes of their homologous sides.

413. *Corollary.* Similar cylinders or cones are to each other as the cubes of the radii of their bases.

414. *Theorem.* Similar polyedrons are to each other as the cubes of their homologous sides.

Proof. Let a polyedron be divided into pyramids by drawing lines from one of its vertices to all its other vertices; any similar polyedron may be divided into similar pyramids by lines similarly drawn from the homologous vertex.

Now these similar pyramids are to each other, by § 412, as the cubes of their homologous sides, or as the cubes of any two homologous sides of the polyedrons; and, from the theory of proportions, their sums, that is, the polyedrons themselves, are to each other in the same ratio, or as the cubes of their homologous sides.

CHAPTER XVIII.

THE SPHERE.

415. *Definition.* A *sphere* is a solid terminated by a curved surface, all the points of which are equally distant from a point within called the *centre.*

416. *Corollary.* The sphere may be conceived to be generated by the revolution of a semicircle, *DAE* (fig. 177) about its diameter *DE.*

417. *Definitions.* The *radius of a sphere* is a straight line drawn from the centre to a point in the

surface ; the *diameter* or *axis* is a line passing through the centre, and terminated each way by the surface.

418. *Corollary.* All the radii of a sphere are equal ; and all its diameters are also equal, and double of the radius.

419. *Theorem.* Every section of a sphere made by a plane is a circle.

Proof. From the centre C (fig. 178) of the sphere draw the perpendicular CO to the section AMB and the radii CA, CM, CB, &c. Since these radii are equal, they must, by § 321, terminate in a circumference AMB, of which O is the centre

420. *Definitions.* The section made by a plane which passes through the centre of the sphere is called a *great circle.* Any other section is called a *small circle.*

421. *Corollary.* The radius of a great circle is the same as that of the sphere, and therefore all the great circles of a sphere are equal to each other.

422. *Corollary.* The centre of a small circle and that of the sphere are in the same straight line perpendicular to the plane of the small circle.

423. *Definition.* The points, in which a radius of the sphere, perpendicular to the plane of a circle, meets the surface of the sphere, are called the *poles of the circle* ; thus P, P' are the poles of AMB.

424. *Corollary.* Since the oblique lines PA, PM, &c. are equally distant from the perpendicular PO, they are equal ; and also the arcs of great circles PA, PM,

&c. are, by § 113, equal; that is, the pole of a circle is equally distant from all the points in the circumference of the circle.

425. *Corollary.* Since the distance *DM* (fig. 177) of a point, in the circumference of a great circle from the pole, is measured by the right angle *DCM*, it is a quadrant.

426. *Scholium.* By means of poles, arcs may be traced upon the surface of a sphere as easily as upon a plane surface.

We see, for example, that by turning the arc *DF* (fig 177) about the point *D*, the extremity *F* describes the small circle *FNG*; and by turning the quadrant *DFA* about the point *D*, the extremity *A* describes the arc of a great circle *AM*.

427. *Theorem.* A point upon the surface of a sphere which is at the distance of a quadrant from each of two other points, is one of the poles of the great circle which passes through these two points.

Proof. Thus, if the distances *DA*, *DM* (fig. 177) are quadrants, the angles *DCA* and *DCM* are right angles, and, therefore, by § 318, *DC* is perpendicular to the circle *AMB*, and its extremity *D* is, by § 423, a pole of the circle *ABM*.

428. *Corollary.* Since the common intersection of two great circles is, by § 420, a diameter, they bisec each other.

429. *Theorem.* Every great circle bisects the sphere

Proof. For if, having separated the two *hemispheres* from each other, we apply the base of one to that of the other,

turning the convexities the same way, the two surfaces must coincide ; otherwise, there would be points in these surfaces unequally distant from the centre.

430. *Definitions.* A *spherical triangle* is a part of the surface of a sphere comprehended by three arcs of great circles.

These arcs, which are called the *sides* of the triangle, are always supposed to be smaller each than a semicircumference. The angles, which their planes make with each other, are the *angles of the triangle.*

Since the sides are arcs, they may be expressed in degrees and minutes, as well as the angles.

431. *Definitions.* A spherical triangle takes the name of *right*, *isosceles*, and *equilateral*, like a plane triangle, and under the same circumstances.

432. *Definition.* A *spherical polygon* is a part of the surface of a sphere terminated by several arcs of great circles.

433. *Definitions.* The portion of a sphere comprehended between the halves of two great circles is called a *spherical wedge*, and the portion of the surface of the sphere comprehended between them is called a *lunary surface*, and is the *base* of the wedge.

434. *Definitions.* A *spherical pyramid* is the part of a sphere comprehended between the planes of a solid angle whose vertex is at the centre.

The *base* of the pyramid is the spherical polygon intercepted by these planes.

435. *Definition.* A plane is *tangent* to a sphere, when it has only one point in common with the surface of the sphere.

Spherical Segment, Sector.

436. *Definitions.* When two parallel planes cut a sphere, the portion of the sphere comprehended between them is called a ***spherical segment***, and the portion of the surface of the sphere comprehended between them is called a *zone*.

The *bases of the segment* are the sections of the sphere, and the *bases of the zone* are the circumferences of the sections.

The *altitude* of the segment or zone is the distance between the sections.

One of the cutting planes may be tangent to the sphere, in which case the zone or segment has but one base.

437. *Definition.* While the semicircle DAE (fig. 177) turning about the diameter DE describes a sphere, every circular sector, as DCF or FCH, describes a solid, which is called a *spherical sector*. The *base* of the sector is the zone generated by the arc DF, or FH.

438. *Theorem.* Either side of a spherical triangle is less than the sum of the other two.

Proof. From the centre O (fig. 179) of the sphere draw the radii OA, OB, OC to the vertices A, B, C of the spherical triangle ABC. The three plane angles AOB, AOC, BOC form a solid angle at O; and each of these angles is, by § 338, less than the sum of the other two. But they are measured by the arcs AB, AC, BC; and, therefore, each of these arcs is less than the sum of the other two.

439. *Theorem.* The sum of the sides of a spherical polygon is less than the circumference of a great circle.

Proof. From the centre O (fig. 180) of the sphere draw the radii OA, OB, OC, &c. to the vertices A, B, C, &c. of the spherical polygon ABC &c. The plane angles AOB, BOC, &c. form a solid angle at O; and the sum of these angles is, by § 339, less than four right angles. The sum of the arcs AB, BC, CD, &c. is, consequently, less than a circumference of a great circ e.

440. *Corollary.* If then, we denote the sides of a spherical triangle by a, b, c, we have

$$a + b + c < 360^\circ.$$

441. *Theorem.* The angle formed by two arcs of great circles is measured by the arc described from its vertex as a pole, and included between its sides.

Proof. The arc AM (fig. 177) measures the angle ACM, which, by § 315, measures the angle of the planes DCA and DCM; and therefore, by § 430, it measures the angle ADM.

442. *Corollary.* The value of the arc AM expressed in degrees, minutes, &c., is the same as that of ADM.

443. *Theorem.* If from the vertices of a given spherical triangle as poles, arcs of great circles are described, another triangle is formed, the vertices of which are the poles of the sides of the given triangle.

Proof. Bet ABC (fig. 181) be the given triangle; let EF, DF, and DE be described, respectively, with A, B, C as poles.

Then, since E is in the arc EF, the distance from E to A is, by § 425, a quadrant; and since E is in the arc DE, the distance from E to C is also a quadrant; and, therefore, by § 427, E is a pole of AC.

In the same way it may be shown, that D is a pole of BC, and F a pole of AB

444. *Definition.* The triangle DEF is called the *polar triangle* of ABC, and in the same way ABC is the polar triangle of DEF.

As several different triangles might be formed by producing the sides DE, EF, and DF, we shall limit ourselves to the one DEF, such that the pole D of BC is on the same side of BC with the vertex A; E is on the same side of AC with the vertex B; and F is on the same side of AB with the vertex C.

445. *Theorem.* If the sides and angles of a spherical triangle and of its polar triangle are expressed in degrees, minutes, &c., the sides of either triangle thus expressed are respectively supplements of the angles of the other triangle.

Proof. Produce the sides AB, AC (fig. 181), if necessary, to G and H.

Since F is the pole of AB, and E the pole of AC, we have, by § 425,

$$EH = FG = 90^\circ.$$

Hence

$$EF = EH + HF = 90^\circ + HF$$

$$GH = GF - HF = 90^\circ - HF,$$

and, therefore,

$$EF + GH = 180^\circ.$$

But, by § 441 and 442,

$$GH = \text{the angle } BAC,$$

whence

$$EF + \text{the angle } BAC = 180^\circ;$$

that is, the side EF and the angle BAC are supplements of each other.

In the same way it may be shown, that DF and the an-

gle ABC, DE and the angle ACB, AB and the angle F, BC and the angle D, AC and the angle E, are respectively supplements of each other.

446. *Corollary.* If therefore we denote the angles of a spherical triangle by A, B, C; and the sides respectively opposite by a, b, c; the angles of the polar triangle must be $180° - a$, $180° - b$, $180° - c$; and the sides of the polar triangle $180° - A$, $180° - B$, $180° - C$.

447. *Theorem.* The sum of the angles of a spherical triangle is greater than two right angles.

Proof. Let A, B, C be the angles of the spherical triangle. The sides of its polar triangle are $180° - A$, $180° - B$, and $180° - C$. Now the sum of these sides, is, by § 440, less than 360°, that is,

$$360° > (180° - A) + (180° - B) + (180° - C)$$

or,

$$360° > 540° - A - B - C,$$

or, by transposition,

$$A + B + C > 540° - 360°,$$

or,

$$A + B + C > 180°;$$

that is, the sum of the angles A, B, C is greater than 180°.

448. *Theorem.* Each angle of a spherical triangle is greater than the difference between two right angles and the sum of the other two angles.

Proof. Let A, B, C be the angles of a spherical triangle; we are to prove that either of these angles, as A, is greater than the difference between 180° and $B + C$.

a. That is, if $B + C$ is less than 180°, we are to prove

$$A > 180° - (B + C)$$

We have, from the preceding proposition,

$$A + B + C > 180°,$$

wnence, by transposition,

$$A > 180° - (B + C).$$

b. But if $B + C$ is greater than 180°, we are to prove

$$A > (B + C) - 180°.$$

Now, of the three sides $180° - A$, $180° - B$, $180° - C$ of the polar triangle, each is, by § 438, less than the sum of the other two ; that is,

$$(180° - B) + (180° - C) > 180° - A$$

or

$$360° - B - C > 180° - A,$$

and, by transposition,

$$A > + B\ C - 360° + 180°,$$

or

$$A > (B + C) - 180°,$$

as we wished to prove.

449. *Theorem.* If two spherical triangles on the same sphere, or on equal spheres, are equilateral with respect to each other, they are also equiangular with respect to each other.

Proof. Let ABC, DEF (fig. 182) be the spherical triangles, of which the sides $AB = DE$, $AC = DF$, and $BC = EF$.

Draw the radii OA, OB, OC, $O'D$, $O'E$ $O'F$ The angles AOB and $DO'E$ are equal, because they are measured by the equal arcs AB and DE ; in the same way, $AOC = DO'F$, $BOC = EO'F$, and therefore, by § 340, the angle of the planes AOB, AOC is equal to that of the planes $DO'E$, $DO'F$, that is, $BAC = EDF$.

In like manner, $ABC = DEF$, and $ACB = DFE$.

450. *Definition.* Two spherical triangles are *symmetrical*, when they are equilateral and equiangular with respect to each other, but cannot be applied to each other, as ABC, ABC' (fig. 183).

451. *Theorem.* If two triangles on the same sphere, or on equal spheres, have a side, and the two adjacent angles of the one respectively equal to a side and the two adjacent angles of the other, they are equal, or else they are symmetrical.

Proof. If the two triangles ABC, DEF (fig. 183) have the side $AB = DE$, the angle $BAC = EDF$, and the angle $ABC = DEF$; the side DE can be placed upon AB, and the sides DF, FE will fall upon AC, BC, or upon the sides AC', BC' of the triangle ABC', symmetrical to ABC.

452. *Theorem.* If two triangles on the same sphere, or on equal spheres, have two sides, and the included angle of the one respectively equal to the two sides and the included angle of the other, they are equal, or else they are symmetrical.

Proof. For one of the triangles may be applied to the other, or to its symmetrical triangle.

453. *Theorem.* In every isosceles spherical triangle the angles opposite the equal sides are equal.

Proof. Let AB (fig. 184) be equal to AC. From A draw AD to the middle of BC.

In the triangles ABD, ACD, the side AD is common, the side $BD = DC$, and the side $AB = AC$; hence, by § 449, the angle $ABC =$ the angle ACB.

454. *Corollary.* Also the angle $ADB = ADC$, and,

therefore, each is a right angle ; and also $DAB = DAC$, that is,

The arc, drawn from the vertex of an isosceles spherical triangle to the middle of the base, is perpendicular to the base, and bisects the angle at the vertex.

455. *Corollary.* An equilateral spherical triangle is also equiangular.

456. *Theorem.* If two angles of a spherical triangle are equal, the opposite sides are also equal, and the triangle is isosceles.

Proof. Let the angle ABC (fig. 184) be equal to the angle ACB. Then let $A'BC$ be the symmetrical triangle, of which $A'B = AB$, and $A'C = AC$.

In the triangles ABC, $A'BC$, the side BC is common , the angle $A'BC = ACB$, for each is equal to ABC ; and the angle $A'CB = ABC$, for each is equal to ACB ; hence, by § 450 and 451, the side $AC = A'B$; and, therefore, $AC = AB$.

457. *Corollary.* An equiangular spherical triangle is also equilateral.

458. *Theorem.* If two spherical triangles on the same, or on equal spheres, are equiangular with respect to each other, they are also equilateral with respect to each other.

Proof. Denote by A, B two spherical triangles which are equiangular with respect to each other ; and by P, Q their polar triangles.

Since the sides of P, Q are, by § 445, the supplements of the angles of A, B ; P, Q must be equilateral with respect to each other ; and, also, by § 449, equiangular with respect to each other But the sides of A, B are, by

§ 445, the supplements of the angles of P, Q, and therefore A, B are equilateral with respect to each other.

459. *Theorem.* Of two sides of a spherical triangle, that is the greater which is opposite the greater angle; and, conversely, of two angles, that is the greater which s opposite the greater side.

Proof. 1. Suppose the angle $C > B$ (fig. 185). Draw CD so as to make the angle $BCD = B$.

Then, by § 455,

$$BD = DC,$$

and $$AB = AD + DB = AD + DC.$$

But, by § 438, $$AD + DC > AC,$$

hence $$AB > AC.$$

2. *Conversely.* Suppose $AB > AC$, the angle C must be greater than B; for if C were equal to or less than B, AB would, by § 456 and the preceding demonstration, be equal to or less than AC.

460. *Theorem.* If, of two sides of a spherical triangle, that which differs most from 90° is acute the opposite angle is acute, and if it is obtuse the opposite angle is obtuse.

Proof. Of the two sides AB, AC (fig. 186) of the spherical triangle ABC, let AC be the one which differs the most from 90°. Produce AB, BC to B'.

Since AB, AB' are, by § 428, supplements of each other, one of them is acute and the other obtuse. Suppose either of them, as AB' to be acute. Take $BH = B'H = 90°$, and take $HC' =$ the difference between AC and 90°. HA is the difference between AB and 90°; therefore

$$HC' > HA$$

a. If, then, AC is acute, we have

$$B'C' = AC, \text{ that is } AC < B'A.$$

Hence, by § 459, in the triangle $AB'C$,

$$\text{the angle } B' < \text{the angle } ACB'.$$

But since B' and B are each equal to the angle of the planes BAB', BCB', they are equal; and, therefore,

$$\text{the angle } B < \text{the angle } ACB'.$$

Again, since AC is acute and AB obtuse,

$$AC < AB;$$

and, in the triangle ABC, by § 459,

$$\text{the angle } B < \text{the angle } ACB.$$

That is, the angle B is less than either the angle ACB or its supplement ACB'; but one of these angles must be acute, and therefore the angle B is acute.

b. If AC is obtuse, we have

$$BC' = AC,$$

that is, $$AC > BA;$$

and, therefore, by § 459,

$$\text{the angle } B > \text{the angle } BCA.$$

Also, as $B'A$ is acute,

$$AC > B'A,$$

and, therefore, by § 459,

$$\text{the angle } B' > \text{the angle } B'CA;$$

that is, the angle B is greater than either the angle ACB or its supplement ACB'; but one of these angles must be obtuse, and therefore the angle B is obtuse.

461. *Corollary.* Of two sides of a spherical triangle, the one which differs most from 90° is opposite the angle which differs most from 90°; and, conversely, of two angles of a spherical triangle, the one which dif-

fers most from $90°$ is opposite the side which differs most from $90°$.

462. *Corollary.* If, of two angles of a spherical triangle, that which differs most from $90°$ is acute, the opposite side is acute; and if it is obtuse, the opposite side is obtuse.

463. *Definition.* If we suppose the surface of the hemisphere to be divided into 360 equal parts, each of these may be called a *degree of spherical surface;* and the degree may be subdivided into 60 *minutes*, and the minute into 60 *seconds*.

464. *Corollary.* Any spherical surface may, then, be expressed by that number of degrees, minutes, &c. which has the same ratio to $360°$, that the given surface has to the hemisphere; it is also measured by an angle of the same number of degrees, minutes, &c.

465. *Theorem.* A lunary surface is measured by double the angle of its bounding circles.

Proof. Let double the angle MAN (fig. 187), expressed in degrees and minutes, be to $360°$, in any ratio as 5 to 48, that is,

$$2\,A : 360° = A : 180° = 5 : 48.$$

Suppose the arcs of great circles $A\,a\,A'$, $A\,b\,A'$, &c to be drawn, so that the angles $MA\,a$, $a\,A\,b$, &c. may be al. equal to each other, and each $\frac{1}{48}$ part of $180°$.

The hemisphere $MAPA'$ is divided into 48 equal lunary surfaces $AMaA'$, $A\,a\,b\,A'$, &c., of which the lunary surface $AMNA'$ contains 5. Hence,

the lunary surface $AMNA'$: the hemisphere $= 5 : 48$

$$= 2\,MAN : 360°,$$

or 2 MAN is, by § 463, the measure of the lunary surface $AMNA'$.

The demonstration is extended to the case in which the angle MAN is incommensurate with 180°, by the principles of § 98.

466. *Theorem.* Two symmetrical spherical triangles are equivalent.

Proof. Let ABC, DEF (fig. 188) be two symmetrical triangles, of which $AB = DE$, $AC = DF$, and $BC = EF$.

Let P be the pole of a small circle passing through the three points A, B, C; then the distances PA, PB, PC must be equal.

Draw DQ making the angle QDE equal to PAB, and draw QE making the angle DEQ equal to ABP. Join QF. In the triangles ABP and QDE the side $DE = AB$, the angle $QDE = PAB$, and $QED = PBA$; and, therefore, by § 451, the side $QD = PA$ and $QE = PB$; and since these triangles are isosceles, they can be applied to each other, and are equal.

In the triangles PAC, QDF, the side $PA = QD$, the side $AC = DF$, and the angle PAC, being the sum of PAB and BAC, is equal to QDF, which is the sum of QDE and EDF; and, therefore, by § 452, the side $QF = PC$; and since these triangles are isoceles, they are equal.

In the same way, it may be proved that the isosceles riangle PBC is equal to QEF.

But the triangle $ABC = PAC + PBC - PAB$,

and the triangle $DEF = QDF + QEF - QDE$.

whence the triangle $ABC =$ the triangle DEF.

467. *Corollary.* Hence all spherical triangles, which

are equilateral or equiangular with respect to each other, are equivalent.

468. *Lemma.* If two spherical triangles have an angle of the one equal to an angle of the other; and the sides which include the angle in one triangle are supplements of those which include it in the other triangle; the sum of the surfaces of the two triangles is measured by double the included angle.

Proof. Let the triangles be ABC and DEF (fig. 189), in which A and D are equal; and AB and AC are respectively supplements of DE and DF.

Produce AB and AC till they meet in A'. ABA' and ACA' are, by § 428, semicircumferences. In the triangles $A'BC$ and DEF, the angles A' and D are equal, being both equal to A; $A'B$ and DE are equal, being supplements of AB; and $A'C$ and DF are equal, being supplements of AC. It follows, therefore, from § 467, that they are equal in surface.

But $A'BC$ and ABC compose the lunary surface $ABCA'$ which is measured by $2\ A$. Therefore the sum of ABC and DEF is also measured by $2\ A$.

469. *Theorem.* The surface of a spherical triangle is measured by the excess of the sum of its three angles over two right angles, or 180°.

Proof. Let ABC (fig. 190) be the given triangle. Produce AC to form the circumference $ACA'C'$, also produce AB and BC to form the semicircumferences ABA' and CBC'.

Then, by § 465,

$$\text{the lunary surface } CABC' = 2\ C,$$
$$\text{the lunary surface } ABCA' = 2\ A,$$

or

$$\text{the surface } ABC + \text{the surface } ABC = 2\ C,$$

$$\text{the surface } ABC + \text{the surface } A'BC = 2\ A;$$

and, by § 468,

$$\text{the surface } ABC + \text{the surface } ABC' = 2\ B,$$

for the sides BC and AB are supplements of BC' and $A\ B$; and the angle ABC is equal to the angle $A'BC'$

The sum of these three equations is

$$3 \times \text{the surface } ABC + \text{the surface } A'BC$$
$$+ \text{the surface } ABC' + \text{the surface } A'BC'$$
$$= 2\ A + 2\ B + 2\ C.$$

But the surface of the hemisphere is, by § 463,

$$\text{the surface } ABC + \text{the surface } A'BC$$
$$+ \text{the surface } ABC' + \text{the surface } A'BC' = 360°;$$

which, subtracted from the previous one, gives

$$2 \times \text{surface } ABC = 2\ A + 2\ B + 2\ C - 360°,$$

or

$$\text{the surface } ABC = A + B + C - 180°.$$

470. *Theorem.* The surface of a spherical polygon is equal to the excess of the sum of its angles over as many times two right angles, as it has sides minus two.

Proof. Let $ABCDEK$ (fig. 191) be the given polygon. Draw from the vertex A the arcs AC, AD, &c., which divide it into as many triangles as it has sides minus two. By the preceding theorem, the sum of the surfaces of all these triangles, or the surface of the polygon, is equal to the sum of all their angles diminished by as many times two right angles as there are triangles; that is, the surface of the polygon is equal to the sum of all its angles diminished by as many times two right angles, as it has sides minus two

Surface described by the revolution of a regular portion of a Polygon.

471. *Theorem.* If a portion $ABCD$ (fig. 192) of a regular polygon, situated entirely upon the same side of a line FG drawn through the centre O of the polygon, revolve about FG as an axis, the surface generated by $ABCD$ has for its measure the product of the circumference inscribed in the polygon by MQ, which is the *altitude* of this surface, or the part of the axis comprehended between the extreme perpendiculars AM, DQ.

Proof. Let I be the middle of AB, OI is the radius of the inscribed circle. Draw IK, BN, CP, perpendicular to FG, and AX perpendicular to BN.

The measure of the surface described by AB is, by § 398, $AB \times$ circumference of which KI is radius, which circumference we will denote by circumf. KI.

The triangles OIK, ABX are similar, since their sides are perpendicular to each other; whence, by § 178 and 234,

$$AB : AX = OI : IK = \text{circumf. } OI : \text{circumf. } IK,$$

or, since $AX = MN$,

$$AB : MN = \text{circumf. } OI : \text{circumf. } IK;$$

and, multiplying extremes and means,

$$AB \times \text{circumf. } IK = MN \times \text{circumf. } OI.$$

Whence the area of the surface described by AB is the product of the circumference of the inscribed circle by the altitude MN.

In like manner the area of the surface described by BC is the product of the circumference of the inscribed circle by the altitude NP; and that described by CD is the product of this circumference by PQ.

Hence the area of the entire surface described by $ABCD$ is the product of the circumference of the in-

scribed circle by the sum of the altitudes MN, NP, PQ; that is, by the entire altitude MQ.

472. *Corollary.* If the axis FG passes through the opposite vertices F, G, the area of the surface described by the semipolygon $FACG$ is the product of the circumference of the inscribed circle by the axis FG.

473. *Corollary.* If the sides of the polygon are infinitely small, the polygon becomes a circle, the entire surface generated is that of a sphere, of which the generating circle is a great circle; and the surface generated by the circular segment $ABCD$ is a zone.

Hence the area of the surface of a sphere is the product of its diameter by the circumference of a great circle.

And, the area of a zone is the product of its altitude by the circumference of a great circle.

474. *Corollary.* Since the area of the great circle is, by § 279, half the product of its radius by its circumference; or one fourth of the product of its diameter by its circumference, it is one fourth of the surface of the sphere; that is

The surface of a sphere is equivalent to four great circles.

475. *Corollary.* If we denote by R the radius of the sphere, by C the circumference of a great circle, by S the surface of the sphere, and by π the ratio of the circumference to the diameter, as in § 237; we have

$$C = 2\,\pi \times R$$

$$S = 2\,\pi \times R \times 2\,R = 4\,\pi \times R^2.$$

476 *Corollary.* If we denote in the same way, by R' and S' the radius and surface of a second sphere, we have

$$S' = 4\,\pi \times R'^2,$$

Solidity of the Sphere.

whence

$$S : S' = 4\pi \times R^2 : 4\pi \times R'^2 = R^2 : R'^2,$$

that is, the surfaces of spheres are to each other as the squares of their radii.

477. *Corollary.* Zones upon the same sphere are to each other as their altitudes; and a zone is to the surface of its sphere as its altitude is to the diameter of the sphere.

478. *Theorem.* The solidity of a sphere is one third of the product of its surface by its radius.

Proof. For the surface of the sphere may be considered as composed of infinitely small planes; and each of these planes may be considered to be the base of a pyramid, which has its vertex at the centre of the sphere, and, consequently, an altitude equal to the radius of the sphere. The sum of the solidities of these pyramids is, then, one third of the product of the sum of their bases by their common altitude, that is, the solidity of the sphere is one third of the product of its surface by its radius.

479. *Corollary.* In the same way, the base of a spherical pyramid or sector may be considered as composed of planes, and, therefore, the solidity of a spherical pyramid or sector is one third of the product of the polygon or zone, which serves as its base, by its radius.

480. *Corollary.* Spherical pyramids or sectors of the same sphere are to each other as their bases; and a spherical pyramid or sector is to the sphere of which it is a part, as its base to the surface of the sphere.

481. *Corollary.* Hence, by § 477, spherical sec-

tors upon the same sphere are to each other as the altitudes of the zones, which serve as their bases; and a spherical sector is to the sphere of which it is a part, as the altitude of its base to the diameter of the sphere.

482. *Corollary.* Denoting by R the radius of the sphere, by S its surface, by V its solidity, and by π the ratio of a circumference to its diameter, we have, by § 475, and 478,

$$S = 4\,\pi \times R^2$$
$$V = \tfrac{1}{3}\,R \times S = \tfrac{4}{3}\,\pi \times R^3.$$

483. *Corollary.* Denoting, in like manner, by R' and V' the radius of another sphere, we have

$$V' = \tfrac{4}{3}\,\pi \times R'^3,$$

whence

$$V : V' = \tfrac{4}{3}\,\pi \times R^3 : \tfrac{4}{3}\,\pi \times R'^3 = R^3 : R'^3;$$

that is, spheres are to each other as the cubes of their radii.

484. *Corollary.* Denoting by R the radius of a sphere, by C the circumference of a great circle, by H the altitude of a zone, by Z the surface of the zone, by V the solidity of its corresponding sector, and using π as before, we have,

$$C = 2\,\pi \times R,$$

from § 473,

$$Z = C \times H = 2\,\pi \times R \times H,$$

and, from § 479,

$$V = \tfrac{1}{3}\,Z \times R = \tfrac{2}{3}\,\pi \times R^2 \times H.$$

485. *Corollary.* The solidity of the spherical segment of one base less than a hemisphere, generated by the revolution of the portion ABC (fig. 193) of

a circle about the radius OA, may be found by subtracting that of the right cone generated by OBC, from that of the spherical sector generated by AOB.

In like manner, the solidity of the spherical segment of one base, greater than a hemisphere generated by the revolution of $AB'C'$, may be found by adding that of the right cone generated by $OB'C'$, to that of the sector generated by AOB'.

486. *Corollary.* The solidity of the spherical segment of two bases generated by $CBB'C'$ (fig. 193), about the diameter AOA', may be found by subtracting that of the segment of one base generated by ABC from that of the segment of one base generated by $AB'C'$.

CHAPTER XIX.

REGULAR POLYEDRONS.

487. *Definitions.* A *regular polyedron* is one, all whose faces are equal regular polygons, and all whose solid angles are equal to each other. These conditions can be fulfilled in only a small number of cases.

a. If the faces are equilateral triangles, polyedrons may be formed of them, having solid angles contained by three of these triangles, by four, or by five. No other polyedron can be formed with equilateral triangles, for six angles of such a triangle are equal to four right angles, and cannot, by § 339, form a solid angle.

b. If the faces are squares, their angles may be arranged by threes. But four angles of a square are equal to four right angles, and cannot form a solid angle.

c. If the faces are regular pentagons, their angles may likewise be arranged by threes.

d. We can proceed no further; for three angles of a regular hexagon are equal to four right angles; three of a heptagon are greater.

488. *Corollary.* There can be only five regular polyedrons; three formed with equilateral triangles, one with squares, and one with pentagons; and in three of these polyedrons each solid angle is formed by three plane angles, and in one of them by four, and in one by five plane angles.

489. *Problem.* To find the number of faces of the regular polyedrons.

Solution. Denote the number of plane angles by which each solid angle is formed by m, and the number of sides of each face by n.

Now it is evident from the symmetrical character of the regular polyedron, that a sphere can be circumscribed about it; and, if the adjacent vertices of the polyedron are joined by arcs of great circles, the surface of the sphere is divided into as many equal regular spherical polygons as the polyedron has faces, and the number of sides of each spherical polygon is n, or the same as that of the face of the polyedron.

Moreover, the number of spherical angles which are formed at each vertex is m; but their sum is equal to that of four right angles, and, since they are equal to each other, each must be represented by 360° divided by m; that is, denoting each spherical angle by A,

Number of Faces of the Regular Polyedrons.

$$A = 360° \div m.$$

Again, the sum of the angles of each spherical polygon is $n \times A$; and therefore the surface of the polygon, which we shall denote by S, is, by § 470,

$$S = n \times A - (n - 2) \times 180°,$$

or $$S = n \times 360° \div m - (n - 2) \times 180°.$$

Hence the number of faces is easily found, and is equal to the number of times which S is contained in the surface of the sphere, or, by § 464 in 720°.

490. *Corollary. When the polyedron is composed of equilateral triangles*, we have $n = 3$, whence

$$S = 1080° \div m - 180°.$$

a. If, then, *the number of plane angles at each vertex is* 3, we have $m = 3$, whence

$$S = 360° - 180° = 180°,$$

which is contained 4 times in 720°, and therefore this polyedron is a *tetraedron.*

b. If the number of plane angles at each vertex is 4, we have $m = 4$, whence

$$S = 270° - 180° = 90°,$$

which is contained 8 times in 720°, and, therefore, this polyedron is an *octaedron.*

c. If the number of plane angles at each vertex is 5, we have $m = 5$, whence

$$S = 216° - 180° = 36°,$$

which is contained 20 times in 720°, and therefore this polyedron is an *icosaedron.*

491. *Corollary. When the polyedron is composed of squares*, we have $n = 4$, and, by § 486, $m = 3$, whence

$$S = 480° - 360° = 120°,$$

which is contained 6 times in 720°, and therefore this polyedron is a *hexaedron* or *cube*.

492. *Corollary. When the polyedron is composed of regular pentagons*, we have $n = 5$, and, by § 486, $m = 3$, whence

$$S = 600^\circ - 540^\circ = 60^\circ,$$

which is contained 12 times in 720°, and therefore this polyedron is a *dodecaedron*

THE END.

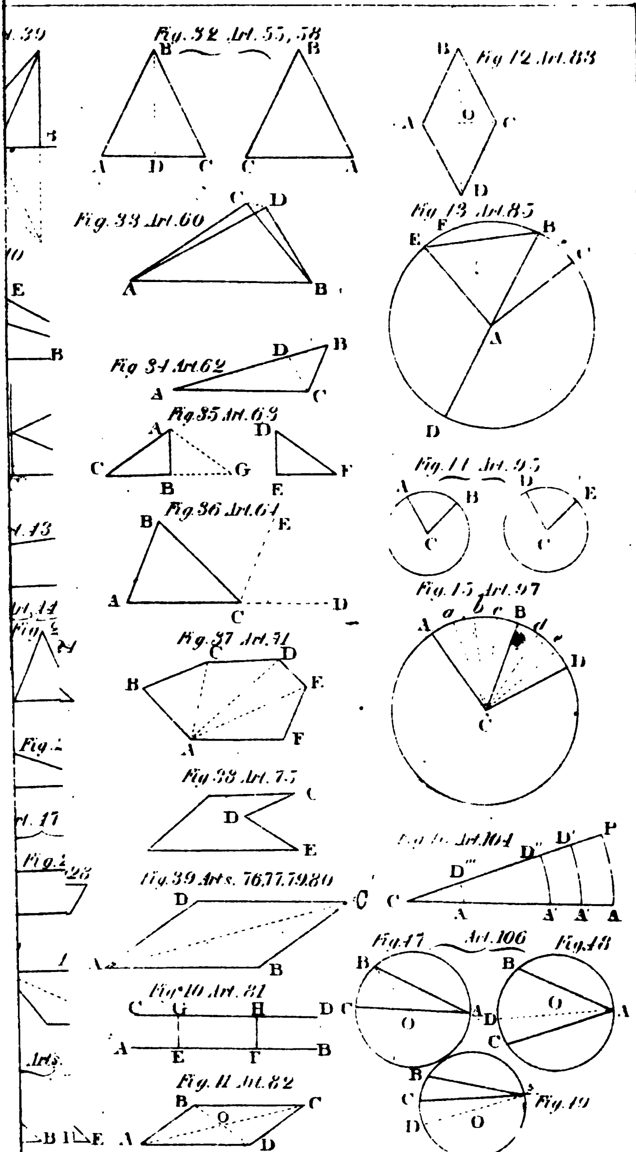


Fig. 32 Art. 55, 58
Fig 12 Art. 83
Fig. 33 Art. 60
Fig 13 Art. 85
Fig 34 Art. 62
Fig. 35 Art. 63
Fig. 14 Art. 95
Fig. 36 Art. 64
Fig. 15 Art. 97
Fig. 37 Art. 71
Fig. 38 Art. 75
Art. 104
Fig. 39 Arts. 76, 77, 79, 80
Fig. 17 Art. 106
Fig. 18
Fig. 40 Art. 81
Fig. 11 Art. 82
Fig. 19

which is contained 6 times in 720°, and therefore this polyedron is a *hexaedron* or *cube.*

492. *Corollary. When the polyedron is composed of regular pentagons,* we have $n = 5$, and, by § 486, $m = 3$, whence

$$S = 600^\circ - 540^\circ = 60^\circ,$$

which is contained 12 times in 720°, and therefore this polyedron is a *dodecaedron*

THE END.

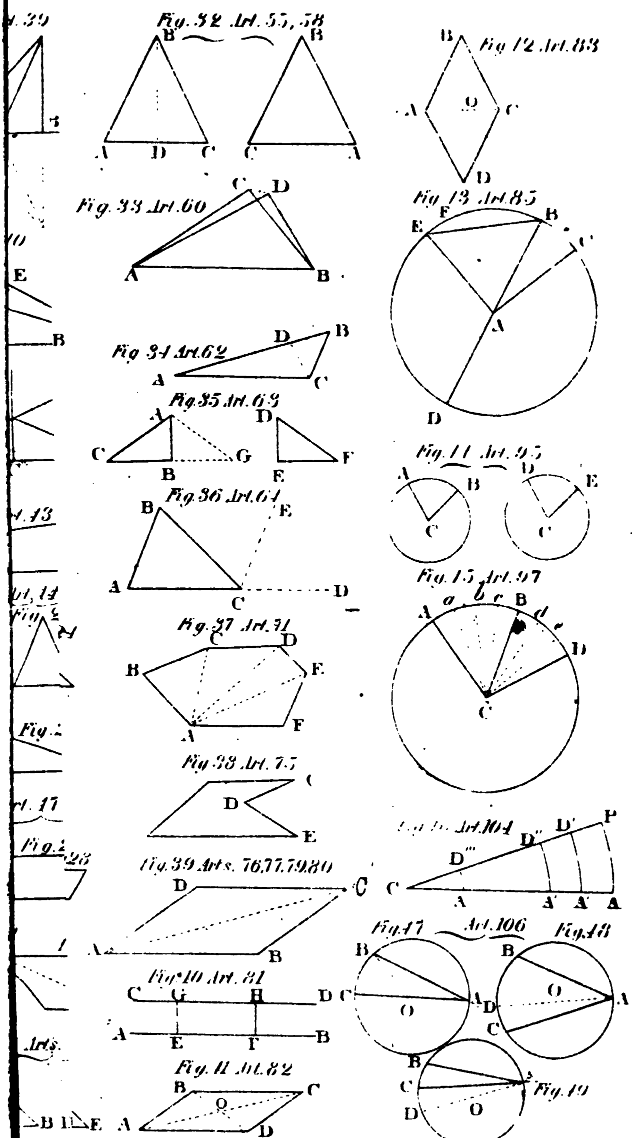


Fig. 32 Art. 55, 58
A D C B C A B
Fig. 12 Art. 83
A B C D O
Fig. 33 Art. 60
A B C D
Fig. 13 Art. 85
E F B C A D
Fig. 34 Art. 62
A B C D
Fig. 35 Art. 63
C A B G D E F
Fig. 14 Art. 95
A B C D E C
Fig. 36 Art. 64
B A C E D
Fig. 15 Art. 97
A a b c B d e D C
Fig. 37 Art. 71
B C D E F A
Fig. 38 Art. 75
D C E
Art. 104
C A A' A' A D''' D'' D' P
Fig. 39 Arts. 76, 77, 79, 80
D C A B
Fig. 47 Art. 106 Fig. 48
B C A O D B O A C
Fig. 40 Art. 81
C G H D A E F B
Fig. 41 Art. 82
B C A O D
B C D O Fig. 49

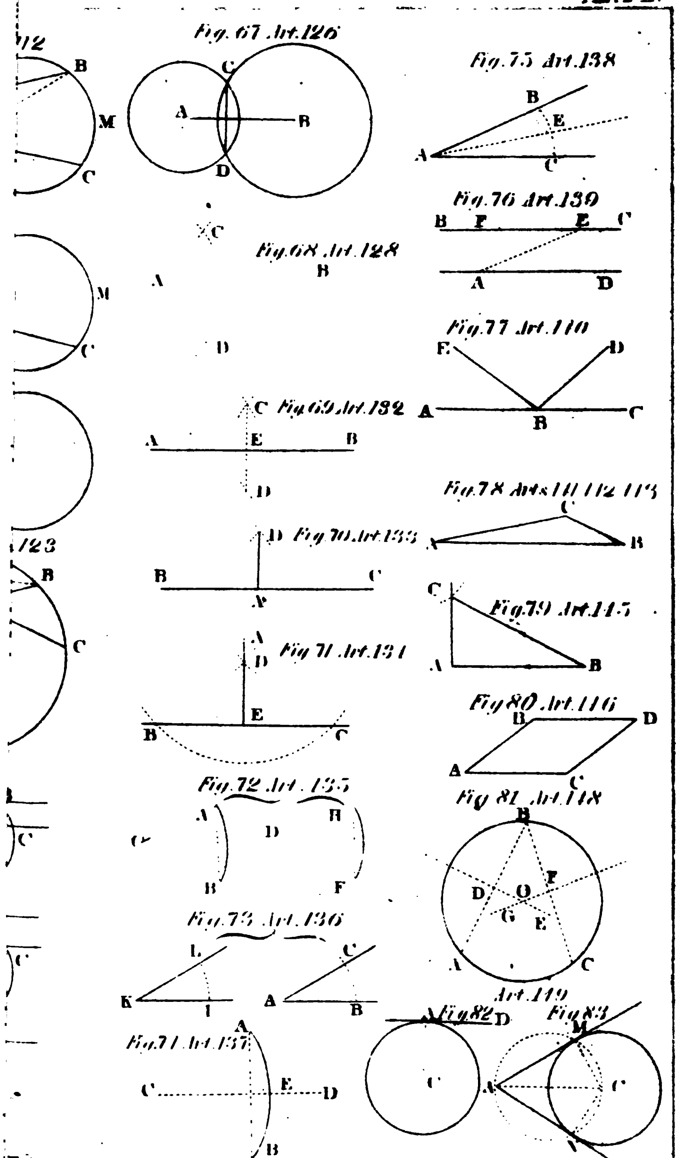
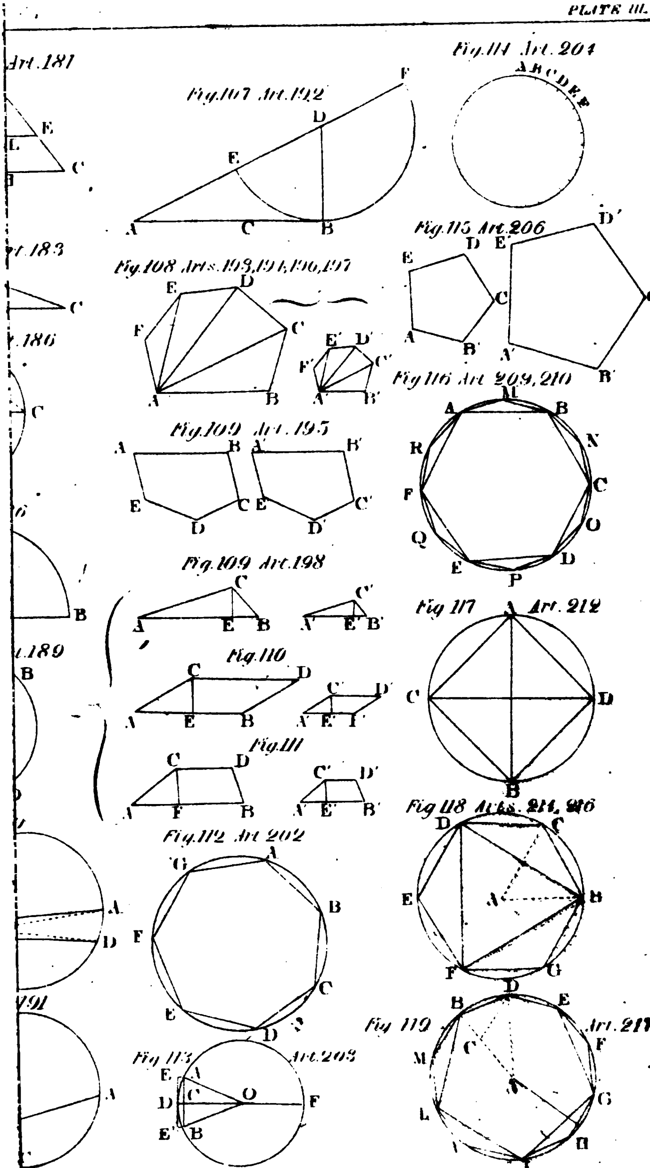


Fig. 67 Art. 126
Fig. 75 Art. 138
Fig. 76 Art. 139
Fig. 68 Art. 128
Fig. 77 Art. 140
Fig. 69 Art. 132
Fig. 78 Arts 141 142 143
Fig. 70 Art. 133
Fig. 79 Art. 145
Fig. 71 Art. 134
Fig. 80 Art. 146
Fig. 72 Art. 135
Fig. 81 Art. 148
Fig. 73 Art. 136
Art. 149
Fig. 82
Fig. 83
Fig. 74 Art. 137

PLATE III.



Art.181
Fig.107 Art.192
Fig.114 Art.204
Art.183
Fig.108 Arts.193,194,196,197
Fig.115 Art.206
Fig.116 Art.209,210
Fig.109 Art.195
Fig.109 Art.198
Fig.117 Art.212
Art.189
Fig.110
Fig.111
Fig.118 Arts.214,216
Fig.112 Art.202
Fig.119 Art.217
Art.191
Fig.113 Art.203

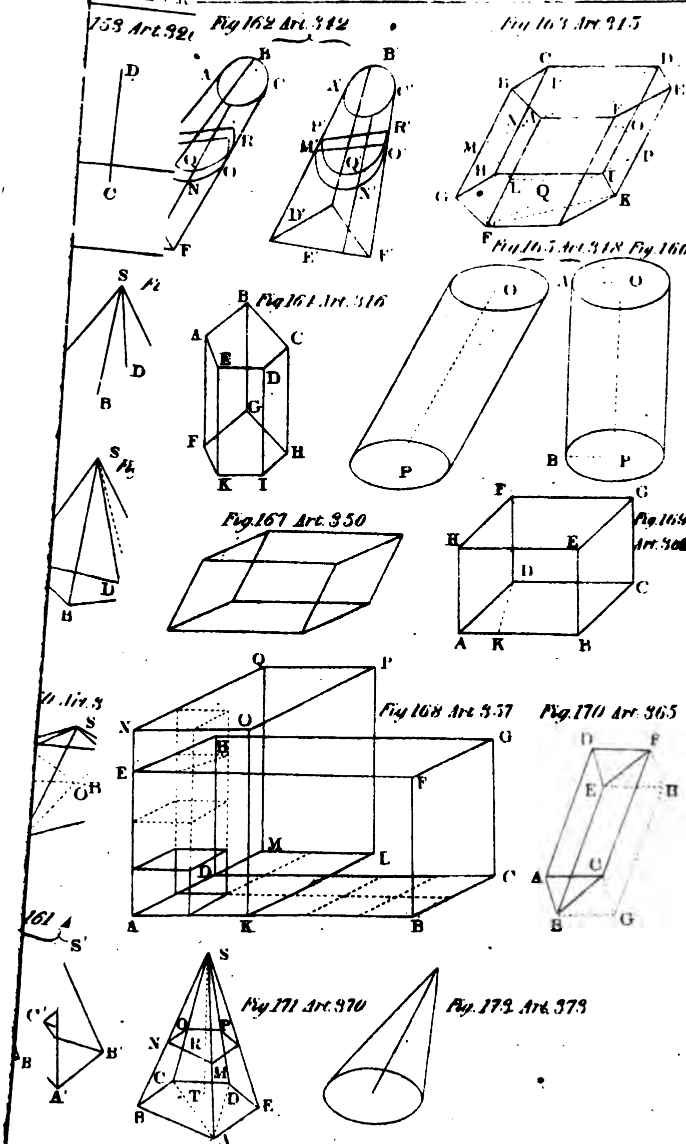


Fig 162 Art 342
Fig 163 Art 315
Fig 164 Art 316
Fig 165 Art 318
Fig 166
Fig 167 Art 350
Fig 169
Fig 168 Art 357
Fig 170 Art 365
Fig 171 Art 370
Fig 172 Art 373

181 Art. 148

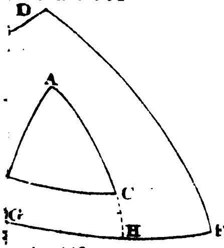
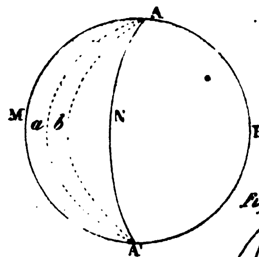


Fig. 187 Art. 164



2 Art. 119

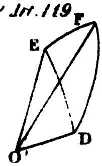


Fig 188 Art 165

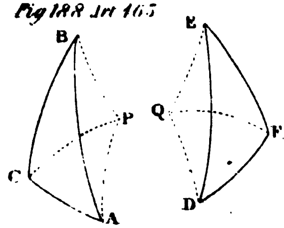
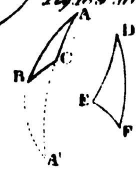
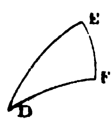


Fig. 189 Art. 167



3 Art 150



190 Art 168

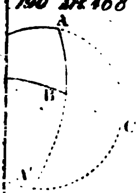
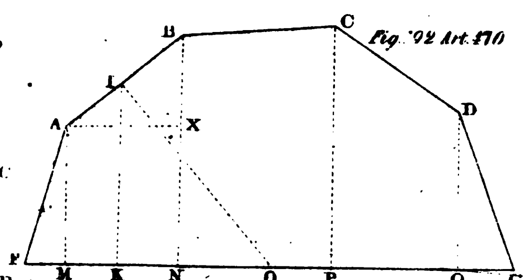


Fig. 192 Art. 170



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191 Art 169

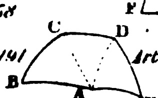
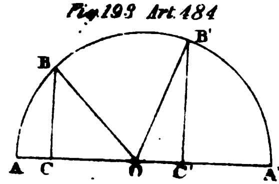


Fig. 193 Art. 484



B'

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